MODELS OF AN ECONOMY

JOHN E. HUTCHINSON

Contents

1. Introduction

We consider mathematical models of a trading economy with both a finite number of agents and a continuum of agents. The latter:

(1) is important as a limit model of the finite case;
(2) is necessary in order to model an economy with a pricing system that cannot be influenced by individual agents;
(3) is necessary if the notions of maximising individual preferences by commodity exchange, and maximising individual preferences by means of some pricing system, are to be equivalent.

Various generalisations are possible. In particular, it is possible to include production.

The main references are [HK88, Hil74].

2. Finite Exchange Economy

A *finite exchange economy* $\mathcal{E} = (\mathbb{R}^\ell_+, \succeq_a, e_a; a \in A) = (\succeq_a, e_a; a \in A)$ is described by the *primitive concepts* of *commodity space*, *set of agents*, *preference relation* and *initial endowment*.

2.1. **Commodity Space.** A *commodity* is a good or service. It is represented by points on $\mathbb{R}_+ = [0, \infty)$. (And so commodities are "infinitely divisible".)

Assume there are ℓ distinct commodities.

A *commodity bundle* is a point in *commodity space* $\mathbb{R}^\ell_+ := \{(x_1, \ldots, x_\ell) : x_i \geq 0 \ \forall i\}$.[1]

2.2. **Set of Agents.** There is a *finite* set A of *agents* (*traders*).

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[1] For points $x, y \in \mathbb{R}^\ell$ we write $x \leq y$ if $x_i \leq y_i \ \forall i$, and write $x < y$ if $x \leq y, x \neq y$.

2.3. **Preference Relations.** There is a *preference relation* (or desirability relation) $\succeq_a$ for each $a \in A$. Thus is a binary relation on the *consumption set* $\mathbb{R}^\ell_+$.[2] A generic preference relation is denoted by $\succeq$.

$x \succeq_a y$ is interpreted as "for a, x is at least as desirable as y". i.e. "x is as good as or better than y".

Unless o.w. stated we assume $\succeq$ is:

- *reflexive*: $x \succeq x$;
- *complete*: $x \succeq y$ or $y \succeq x$ (possibly both);
- *transitive*: $x \succeq y$ and $y \succeq z \Longrightarrow x \succeq z$;
- *continuous*: $\{x : x \succeq y\}$ and $\{x : y \succeq x\}$ are closed $\forall y \in \mathbb{R}^\ell_+$.

Assuming just reflexivity and transitivity we define

(1) x is *as equally desirable as* y, written $x \sim y$, by $x \succeq y$ and $y \succeq x$ (x and y lie on the same indifference set for a);
(2) x is *preferred to* (more desirable than) y, written $x \succ y$, by $x \succeq y$ and not $x \sim y$.

Continuity is then equivalent to requiring that if $x \succ y$ then for all u in some nbd of x and all v in some nbd of y, $u \succ v$.

It is often convenient to represent $\succ$ by a *Pareto utility function* $u : \mathbb{R}^\ell_+ \to \mathbb{R}$, where $u(x) > u(y)$ iff $x \succ y$. Indifference sets are represented by level sets of u. See Figure 1.

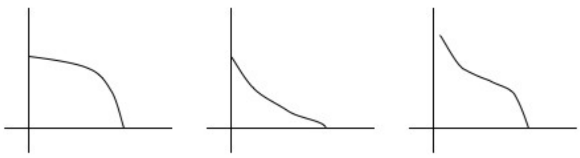

FIGURE 1. Possible level sets of $\succ$.

The following are further restrictions on the preference relations:
Monotonicity: $y \geq x$ and $y \neq x \Longrightarrow y \succ x$. ("more is better")
Convexity: $\{y : y \succeq x\}$ to be convex for all $x \in \mathbb{R}^\ell_+$.
Strict convexity: $x \sim y,\ x \neq y,\ 0 < \lambda < 1 \Longrightarrow \lambda x + (1-\lambda)y \succ y$.

2.4. **Initial Endowments.** The *initial endowment* (*allocation*) for a is an element $e_a \in \mathbb{R}^\ell_+ \setminus \{0\}$.

Every agent has non zero initial endowment as o.w. there is nothing to trade and he will not be a part of the economy.

3. Redistribution in an Economy

3.1. **Defined concepts.** Define the following for an economy $\mathcal{E} = (\succeq_a, e_a; a \in A)$:

(1) A *coalition* S is a *non empty* subset of A.
(2) An *allocation* is a map $f : A \to \mathbb{R}^\ell_+$. It is a *redistribution* (*feasible allocation*) if $\sum_a f(a) = \sum_a e(a)$. (Conservation of commodities.)
(3) A coalition S can *improve upon* a redistribution f (using its *original* endowment $\sum_{a\in S} e(a)$) if there is a redistribution g such that
 (a) $g(a) \succ f(a)$ for all $a \in S$, (every agent in S has an improved situation)
 (b) $\sum_{a\in S} g(a) = \sum_{a\in S} e(a)$.
 W.l.o.g. one can take $g(a) = f(a)\ \forall a \notin S$, i.e. only those resources in S are redistributed.

3.2. **Edgeworth Box.** The "Edgeworth Box" in Figure 2 represents two traders and two commodities. One can do this in $\mathbb{R}^\ell$ for two traders and ℓ commodities.

The origin for the first is in the lower left corner, the indifference lines are solid, and the initial endowment is e_1. The origin for the second is in the upper right corner, the indifference lines are dotted, and the initial endowment is e_2. The total endowment is $\overline{e} = e_1 + e_2$ and so the initial endowment for *both* traders is represented by the *point* e_1 with origin lower left corner

[2]It would be more realistic to restrict to a convex subset X of $\mathbb{R}^\ell_+$, which may depend on a and represents the set of commodity bundles sufficient for the agent a to survive. The theory is essentially unchanged.

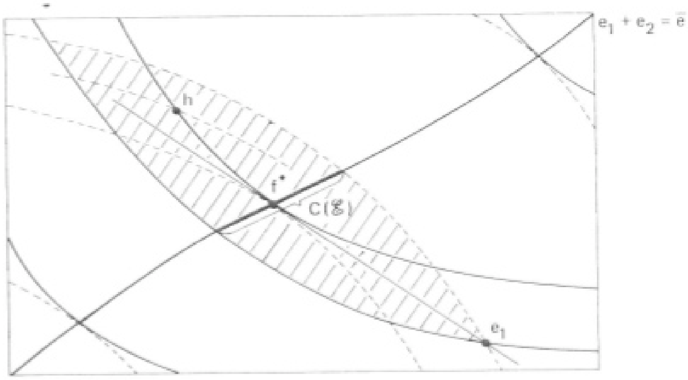

FIGURE 2. Two traders with two commodities. See [Hil82, p835].

— this being the same as the point e_2 with origin upper right corner. Each point in the box corresponds to a redistribution.

The coalition consisting of just the first trader can improve upon any allocation below the indifference line through e_1. The coalition consisting of just the second trader can improve upon any allocation above the dotted indifference line through e_1. This leaves the shaded lens region. In here the coalition of both traders can improve upon any allocation which is not in the set of points $\mathbb{C}(\mathcal{E})$ where indifference lines are tangential.

3.3. **Core of an economy.** The *core* $\mathbb{C}(\mathcal{E})$ of the economy $\mathcal{E} = (\succeq_a, e_a; a \in A)$ is the set of all redistributions that no coalition can improve upon in the previous sense.[3]

That is, a redistribution $f \in \mathbb{C}(\mathcal{E})$ iff no coalition can improve upon f by redistributing its *initial* endowment. More precisely, a redistribution $f \in \mathbb{C}(\mathcal{E})$ iff there is *no* coalition S and redistribution g such that

$$\forall a \in S \ \big(g(a) \succ f(a)\big) \quad \text{and} \quad \sum_{a\in S} g(a) = \sum_{a \in S} e(a). \tag{1}$$

4. Prices in an Economy

Consider an economy $\mathcal{E} = (\succeq_a, e_a; a \in A)$. We introduce an additional concept.

4.1. **Pricing System.** A *pricing system* or *price vector* in $\mathcal{E}$ is a vector $p \in \mathbb{R}^\ell_+$. (Usually $p >> 0$, i.e. $p_i > 0 \ \forall i$). The interpretation is that $p \cdot x = \sum_\ell p_\ell x_\ell$ is the price of the commodity bundle $x \in \mathbb{R}^\ell_+$. (We assume monotonicity.)

The *budget set* for $a \in A$, corresponding to the pricing system p and initial endowment e_a, is

$$\beta(e_a, p) = \{x \in \mathbb{R}^\ell_+ : p \cdot x \le p \cdot e_a\}.$$

It is interpreted as the set of commodity bundles that a can afford within his "budget" as given by p and e_a.

The *income* of a is $p \cdot a$.

The *demand set* for $a \in A$ is

$$\begin{aligned}\phi(\succeq_a, e_a, p) &= \{x \in \beta(e_a,p) : \nexists y \in \beta(e_a,p) \text{ s.t. } y \succ x\}\\ &= \{x \in \beta(e_a,p) : y \in \beta(e_a,p) \Longrightarrow x \succeq y\}\\ &= \{x \in \mathbb{R}^\ell_+ : p\cdot x = p \cdot e_a \quad \& \quad y \succ x \Longrightarrow p \cdot y > p \cdot x\}.\end{aligned}$$

It is the set of commodity bundles that a cannot improve upon within his budget, and is determined by his preferences, initial endowment and the pricing system. In other words, it is the set of "most preferred" bundles available to a within his budget.

The demand set is a singleton if the preference set is strictly convex and $p >> 0$.

[3]If such an allocation f is proposed, no coalition will be able to improve on f by redistributing its *original* endowment. ***But could S improve the situation of all its members by redistributing the endowment obtained from f itself?***

4.2. **Walras Equilibrium.** A *Walras equilibrium* for $\mathcal{E}$ is a redistribution f and a price system p such that, for every $a \in A$, $f(a)$ is in the demand set $\phi(\succ_a, e_a, p)$.

More precisely,

$$f(a) \in \phi(\succ_a, e_a, p) \quad \forall a \in A, \qquad \sum_{a\in A} f(a) = \sum_{a \in A} e(a). \tag{2}$$

In other words, (f, p) is a Walras equilibrium iff no agent a can improve upon $f(a)$ with the prevailing price system and his budget. That is,

$$\forall a \in A \quad x \succ_a f(a) \Longrightarrow p \cdot x > p \cdot f(a). \tag{3}$$

Alternatively, a Wallas equilibrium is a redistribution and a price system such that *total demand equals total supply*. One also call this a *price equilibrium* or *competitive equilibrium*.[4] A Walras price system *decentralises the redistribution problem*.

The redistribution f is called a *Walras allocation* and the set of Walras allocations is denoted by $\mathbb{W}(\mathcal{E})$.

The price vector p is called a *Walras price vector* or *equilibrium price vector* and the set of equilibrium price vectors is denoted by $\Pi(\mathcal{E})$.

In Figure 2 the (unique) Walras equilibrium for the initial endowment e_1 is (f^*, p), where p is normal to the line through e_1 and f^*.

Proposition 4.1. *$\mathbb{W}(\mathcal{E}) \subset \mathbb{C}(\mathcal{E})$. (*No *assumptions on the structure of "$\succ$" here.)*

Proof. Assume $f \in \mathbb{W}(\mathcal{E})$ with equilibrium price vector p. Then by (3) $\forall a \in A$,

$$x \succ_a f(a) \Longrightarrow p \cdot x > p \cdot f(a). \tag{4}$$

Assume $f \notin \mathbb{C}(\mathcal{E})$. Then by (1) $\exists$ a redistribution g s.t.

$$\forall a \in S \ \big(g(a) \succ_a f(a)\big) \quad \text{and} \quad \sum_{a \in S} g(a) = \sum_{a\in S} e(a). \tag{5}$$

If $a \in S$, one has from the first part of (5) that $g(a) \succ_a f(a)$, and hence from (4) that $p \cdot g(a) > p \cdot e(a)$. Hence $p \cdot \sum_{a\in S} g(a) > p \cdot \sum_{a \in S} e(a)$.

But from the second part of (5), $p \cdot \sum_{a \in S} g(a) = p \cdot \sum_{a\in S} e(a)$.

This contradiction implies the second assumption is false and so $\mathbb{W}(\mathcal{E}) \subset \mathbb{C}(\mathcal{E})$. ☐

References

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[Hil74] ———, *Core and equilibria of a large economy*, Princeton University Press, 1974. Princeton Studies in Mathematical Economics, No. 5.

[4]This is misleading terminology as there is not any competition involved.