FOUNDATIONS OF MATHEMATICS

History, Mathematics, Philosophy, Personalities, 1870 – present

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Almost complete version

OUTLINE

Suggestions

- Don't read the slides closely, they are available online;
- except the blue phrases which I will elaborate on, and the red laser pointer.
- Don't sweat the few technical slides, all slides stand independently.
- Curated and annotated bibliography at the end.

- Thanks to Jim Borger and Yiming Xu.
  I became interested in putting this material together after talking to them about their logic course.

OUTLINE

1870–1900 Controversies and Foundational Attempts

- Cantor (1874–1897)
  - infinite sets, cardinals, ordinals, origins of modern set theory
  - motivated by his work in analysis and algebra,
  - extremely controversial.
- Frege(1879):
  - First approximation to modern predicate logic,
  - Versions of logical connectives ($\neg$, $\rightarrow$), quantifiers ($\forall$, $\exists$).
- Peano (1889)
  - Formalised arithmetic, including induction,
  - but not the logic.
- Hilbert (1899)
  - Book: "Foundations of Geometry" [Hil71]
  - Axiomatic approach, notion of proof
  - Syntactic notions: consistency, independence of axioms.
  - Semantic notions: interpretation, truth.

Cantor: Sizes of Infinity

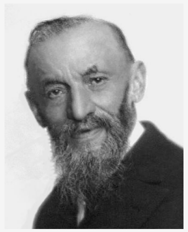
$$\mathbb{N} \sim \mathbb{Z} \sim \mathbb{Q} \not\sim \mathbb{R}$$

natural numbers $\sim$ integers $\sim$ rationals $\not\sim$ reals

Cantor, Frege


- Georg Cantor (1845–1918) Germany, Halle university. Number theory, analysis, set theory. Suffered from depression.
- Gottlob Frege (1848–1925) Germany, Jena. Highly introverted. Foundations of mathematics, philosophy.
  - "Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell, just as the printing of this volume was nearing completion."

Peano, Hilbert




- Giuseppe Peano (1858–1932) Italy, Turin. Logic, analysis, set theory, number theory.
- David Hilbert (1862–1943) Göttingen.
  - 5 periods: algebra, foundations (1897-2003), analysis and PDE's, math physics, foundations (1917-1930).
  - In the 1930's Göttingen maths department purged by the Nazis, forcing out all Jews and those married to a Jew.
  - Hilbert, isolated, remains behind.
  - Effectively disowned his son who suffered mental illness.

OUTLINE

Hilbert's 23 Problems

- Hilbert 1900: 23 problems at the International Congress of Mathematicians.
  - 1. Cardinality between set of rationals and set of reals?
  - 2. Consistency of axioms for real number system, set theory?
  - 10. Diophantine equations, decidability?
- General problem: make precise the ideas behind and the foundations for (in increasing order of difficulty)
  - arithmetic,
  - real number system,
  - theory of sets.

Logicism v. Axiomatism

- Frege and Russell: Logicism – Mathematics as a development of logic.
- Hilbert: Axiomatism/Formalism – branches of mathematics as consequences of various sets of axioms. Axioms motivated by physical and intellectual considerations.
- Axiomatic approach was controversial and opposed to logicism, but eventually becomes dominant.

- Russell (1902): Discovers flaw in Frege's work.
  - Frege assumes: for every property $P(x)$ there is a set y of all x with that property, i.e. $y = \{x \mid P(x)\}$.
  - Russell asks: What if "$P(x)$" is "$x \notin x$"?
    - If $y \in y$ then $y \notin y$,
    - if $y \notin y$ then $y \in y$.

    Contradiction either way.

Principia Mathematica 1

(1910–13) Russell & Whitehead's: Principia Mathematica

- 3 volumes, >1900 pages, dense and obscure notation.
- Uses logicism approach and (ramified) type theory.
- Treats real number system, set theory, measure theory.
- Types: For example power set hierarchy $X, \mathcal{P}(X), \mathcal{PP}(X), \ldots$.
- Ramified types with a notion of reducibilty relate (e.g.) subsets of X defined at a higher level back to a type at the first level.
- "Mathematical" axioms of infinity, choice and reducibility (essentially axiom of replacement), *are* needed, but then *ramified* types are *not* needed.
- Influential ≈ 20 years, then replaced by axiomatic approach.
- Type theory now fundamental in proof theory, comp. science

Principia Mathematica 2

(Unfair Comment): After 350 pages of Vol I in Principia Mathematica, progress towards $1 + 1 = 2$.

$*54{\cdot}43.\ \vdash :.\ \alpha, \beta \in 1 \,.\, \supset :\ \alpha \cap \beta = \Lambda \,.\, \equiv .\ \alpha \cup \beta \in 2$

Dem.

$\vdash .\ *54{\cdot}26 \,.\, \supset \vdash :.\ \alpha = \iota`x \,.\, \beta = \iota`y \,.\, \supset :\ \alpha \cup \beta \in 2 \,.\, \equiv .\ x \neq y \,.$

$[*51{\cdot}231] \qquad \equiv .\ \iota`x \cap \iota`y = \Lambda \,.$

$[*13{\cdot}12] \qquad \equiv .\ \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash .\ (1) \,.\, *11{\cdot}11{\cdot}35 \,.\, \supset$

$\vdash :.\ (\exists x, y) \,.\, \alpha = \iota`x \,.\, \beta = \iota`y \,.\, \supset :\ \alpha \cup \beta \in 2 \,.\, \equiv .\ \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash .\ (2) \,.\, *11{\cdot}54 \,.\, *52{\cdot}1 \,.\, \supset \vdash .$ Prop

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

After a further 400 pages in Vols I & II, completion of argument $1 + 1 = 2$, a result claimed by Russell to be occasionally useful:

$*110{\cdot}643.\ \vdash .\ 1 +_c 1 = 2$

Dem.

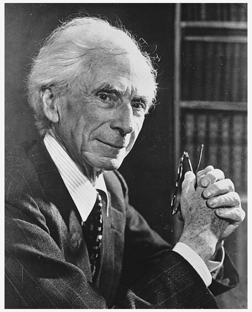
$\vdash .\ *110{\cdot}632 \,.\, *101{\cdot}21{\cdot}28 \,.\, \supset$

$\vdash .\ 1 +_c 1 = \hat{\xi}\{(\exists y) \,.\, y \in \xi \,.\, \xi - \iota`y \in 1\}$

$[*54{\cdot}3] \quad = 2 \,.\, \supset \vdash .$ Prop

The above proposition is occasionally useful. It is used at least three times, in $*113{\cdot}66$ and $*120{\cdot}123{\cdot}472$.

Russell



- Bertrand Russell (1872–1970) UK. mathematics, logic, philosophy, peace activist, public intellectual, 1950 Nobel Prize in Literature.
- "Nature and books and (later) mathematics saved me from complete despondency" (autobiography)

Discouraging Aftermath

- Skolem[Sko22]: "Russell and Whitehead ... constructed a system of logic that provides a foundation for set theory; if I am not mistaken, however, mathematicians have taken little interest in it".
- Hilbert and Ackermann [HA50, pps 153,163] "We no longer have any reason to consider this theory in any more details ... their discussion is unnecessarily complicated".
- *A Mathematician's Apology* [Har40], Russell tells Hardy of his horrible dream: "I was dreaming about the great library of the world, and I saw someone going around with a wheelbarrow clearing out rubbish. . . . I saw him carting off the last surviving copy of Principia Mathematica."

Set Theory, Modern Beginnings

- Zermelo [Zer08]: Set theory axioms (ZF/ZFC)
  - informal separation axiom using "definite propositions"
  - Proved: A.C. $\Leftrightarrow$ sets can be well ordered
  - *Jerry Bona 1977: "the axiom of choice is obviously true, the well-ordering principle is obviously false, and Zorn's Lemma – who knows?"*
- Skolem [Sko22]
  - Radical new idea: a set as an object in a collection (i.e. domain) related to other objects by a binary relation "ϵ".
  - The relation "ϵ" has properties which mirror the intuitive properies of sets.
  - Gives first order axiomatisation of a second order notion
  - Provides precise separation and replacement axioms
  - Proves existence of countable models of ZFC.

Zermelo, Skolem

Zermelo, Skolem

- Ernst Zermelo (1871–1953) Zurich, Freiburg. 1935: Resigned his Freiburg chair in objection to the Nazi regime.
- Thoralf Skolem (1887–1963) Norway, Oslo. Some time in Göttingen, mostly researched independently but active in Norwegian mathematics community.

Hilbert's Program

- Hilbert's Program 1917–1930, @ Göttingen
- Principles of Mathematical Logic Hilbert, Bernays, Ackermann (1928) (second edn 1938) [HA50]
  - First textbook exposition of first-order logic,
  - Posed problems of completeness and decidability
- (1929) Gödel's Completeness Theorem: good news for Hilbert.
  - A sentence ϕ of first-order logic is provable from axioms T — using the rules and and logical axioms of first order logic — iff it is true in all models of T, i.e.
    $$T \vdash \phi \quad \Longleftrightarrow \quad T \models \phi$$
    (Think of group axioms.)
  - (Discussion in subsequent slides)
  - Gödel's bad news, the Incompleteness Theorems, came in 1931.

Intermission: First Order Logic

- Quantifiers range over elements, not subsets.
- Typical Example: A first order set T of axioms for groups is
  - $\forall x \forall y \forall z \left(x \cdot (y \cdot z) = (x \cdot y) \cdot z\right)$ (associativity)
  - $\forall x \left((x \cdot e = x) \wedge (e \cdot x = x)\right)$ (identity)
  - $\forall x \exists y \left((x \cdot y = e) \wedge (y \cdot x = e)\right)$ (inverse)
- Propositional connectives: $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\rightarrow$ (implies).
- Quantifiers: $\forall$ (for all), $\exists$ (there exists).
- Interpretation: Group $\mathcal{G} \equiv \langle G, \cdot^{\mathfrak{G}}, e^{\mathfrak{G}} \rangle$
  - G is the underlying set
  - "$=$" has standard intepretation ("is the same element in G as")
  - Multiplication "$\cdot$" interpreted by function $\cdot^{\mathfrak{G}} : G \times G \to G$
  - Identity symbol "e" interpreted by $e^{\mathfrak{G}} \in G$

Gödel's Completeness Theorem 1

Theorem

A sentence ϕ of first-order logic is provable from axioms T iff ϕ is true in all models of T, i.e. $T \vdash \phi \Leftrightarrow T \models \phi$.

Proof Idea.

($\Rightarrow$) $\quad T \vdash \phi \Rightarrow T \models \phi$ is essentially just that the logical axioms and rules of inference preserve truth, are sound.

($\Leftarrow$) $\quad$ Suppose $T \nvdash \phi$.

Let $\perp$ be any logically false sentence such as $\psi \wedge \neg\psi$.

From the logical rules of inference it follows

$T \cup \{\neg\phi\} \nvdash \perp$, i.e. $T \cup \{\neg\phi\}$ is consistent,

From any consistent set of sentences one can build a model by iteratively adding sentences or their negation to maintain consistency, and new constant symbols for existential formulae to witness their validity. (Henkin's argument [Coh06, Chapter 1.4])

Starting from $T \cup \{\neg\phi\}$, this gives a model whose universe corresponds to the set of new constant symbols and in which T is true and ϕ is false. So $T \not\models \phi$. □

Gödel's Completeness Theorem 2

Corollary

A set T of axioms has a model iff T is syntactically consistent. i.e. iff $T \nvdash \bot$.

Corollary (Compactness)

A set T of axioms has a model iff every finite subset of T has a model.

Proof.

Every finite $F \subset T$ has a model

$\iff$ for every finite $F \subset T$, $F \nvdash \bot$ (previous corollary)

$\iff$ $T \nvdash \bot$ (proofs are finite)

$\iff$ T has a model (previous corollary) □

Skolem 1930's, Łoś (1955): now standard proof of compactness which avoids syntactic notions and combines the models of all finite sentences via an ultraproduct construction.

Peano Arithmetic Formalised

Peano Arithmetic [PA] has first order axioms

- $\forall x(Sx \neq 0)$; $Sx = Sy \to x = y$; (S is successor)
- $x + 0 = x$; $x + Sy = S(x+y)$; (addition)
- $x \cdot 0 = 0$; $x \cdot Sy = x \cdot y + x$; (multiplication)
- $\Big(\phi(0) \wedge \forall x(\phi(x) \to \phi(Sx))\Big) \to \forall x \phi(x)$ (induction)

($\phi(x) = \phi(x, y_1, \ldots, y_n)$ any first order formula with parameters.)

We consider formulae/expressions built up using the language $\mathcal{L}$ of arithmetic ,
i.e. using $S, +, \cdot, =$, propositional connectives, and quantifiers.

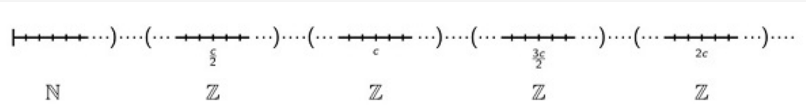
Define $x < y$ by $\exists z(z \neq 0 \wedge y = x + z)$, follows $<$ is linear.

But are the (induction) axioms consistent?
Is there a finite constructive proof of consistency?
Difficulties lie in the complexity of the ϕ.

Nonstandard Models

- Start with axioms [PA].
- Add a new constant symbol c and axioms
  $c > 0$, $c > 1$, $c > 2$, ...
- By compactness, there is a new, very different, model of PA!
- There are uncountably many non-isomorphic, but countable, such models!!

The order type of all such models is the same:
order type of "$<$" is $\mathbb{N}$ followed by $\mathbb{Q}$ copies of $\mathbb{Z}$;

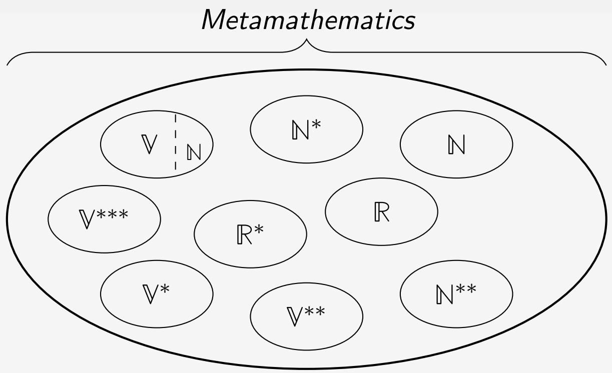


Usually written $\mathbb{N} + \mathbb{Z} \times \mathbb{Q}$.

Metamathematics 1

- Metamathematics : the mathematical study of mathematical systems.
  the theory of mathematical theories.
  usually identified with mathematical logic.
- Frege's work, Russell and Whitehead's Principia, Hilbert's axiomatic approach, Gödel's completeness and incompleteness theorems, Gentzen's consistency result, are metamathematical examples.
- The differing assumptions of Metamathematics may:
  - be constructive: proof of Gödel incompleteness theorems;
  - involve simple notions of "potential" infinity as in Gödel's completeness theorem and Gentzen's consistency result;
  - involve all of ZFC (or more): discussing a model of arithmetic within a model of set theory.

Metamathematics 2



- $\mathbb{N}$ the standard model of arithmetic, $\mathbb{N}^*$ and $\mathbb{N}^{**}$ nonstandard
- Similarly for models $\mathbb{V}, \mathbb{V}^*, \mathbb{V}^{**}, \mathbb{V}^{***}$ of ZFC.
- $\mathbb{V}, \mathbb{V}^*, \mathbb{V}^{**}, \mathbb{V}^{***}$ each contain a model of arithmetic — discussion of which is metametamathematics.
- Different models $\mathbb{R}$ and $\mathbb{R}^*$ — nonstandard models of analysis. See later.

OUTLINE

1. 1870–1900 FOUNDATIONAL CRISES
2. 1900–1930 HILBERT & THE GÖTTINGEN SCHOOL
3. 1930's GÖDEL DESTROYS, GENTZEN REBUILDS
   Gödel: No Effective Axiomatisation for Arithmetic, Real Number System, Set Theory
   Gödel: No Constructive Proof of Consistency
   Gentzen Invents Modern Proof Theory
   Gentzen's Consistency Proof
4. 1936 COMPUTABILITY @ PRINCETON
5. 1942-1970:TARSKI & THE BERKELEY SCHOOL
6. 1939,1963 AXIOM CHOICE & CONTINUUM HYPOTHESIS
7. PROOF THEORY @ STANFORD, 1955–

Gödel's First Incompleteness Theorem (1931)

THEOREM: No set of axioms T is complete for the true sentences of arithmetic, or for real number system, or for set theory.

PROOF SKETCH

- Suppose T is [PA] or some *computable* extension in $\mathcal{L}$.
  Then there are *computable* listings of all sentences in $\mathcal{L}$, all formulae in $\mathcal{L}$, all proofs from T, etc.
- $\phi_n(v)$ denotes n-th formula of $\mathcal{L}$ with the one free variable v.
  $\mathscr{P}rf_T(m,n)$ holds iff the m-th proof shows $T \vdash \phi_n(n)$
  Then $\mathscr{P}rf_T(m,n)$ is *computable*.
- There is a formula $\mathrm{Prf_T}(v,w)$ in $\mathcal{L}$ "mirroring" $\mathscr{P}rf_T(m,n)$:
  $\forall\, m,n \qquad \mathscr{P}rf_T(m,n)$ iff $\mathbb{N} \vDash \mathrm{Prf_T}(m,n)$.
  So: $\quad \forall n \qquad T \nvdash \phi_n(n)$ iff $\mathbb{N} \vDash \neg\exists w\ \mathrm{Prf_T}(w,n)$.
- $\neg\exists w\ \mathrm{Prf_T}(w,v) \equiv \phi_k(v)$ for some computable k.
  So: $\quad \forall n \qquad \mathbb{N} \vDash \phi_k(n)$ iff $T \nvdash \phi_n(n)$.
  $\therefore \qquad \mathbb{N} \vDash \Phi_T$ iff $T \nvdash \Phi_T$,
  where $\Phi_T := \phi_k(k)$, $\quad \Phi_T$ says: "I am not provable from T"

Gödel's First Incompleteness Theorem (continued)

- Recall $\qquad \mathbb{N} \vDash \Phi_T$ iff $T \nvdash \Phi_T$.
- Theorem: $\quad \mathbb{N} \vDash \Phi_T,\ T \nvdash \Phi_T,\ T \nvdash \neg\Phi_T$.
  (Proof: Assume $T \vdash \Phi_T$. Then $\mathbb{N} \nvDash \Phi_T$ by "Recall" and $\mathbb{N} \vDash \Phi_T$ by "axiom soundness". Contradiction. Hence $T \nvdash \Phi_T$.
  Hence $\mathbb{N} \vDash \Phi_T$ by "Recall".
  Assume $T \vdash \neg\Phi_T$. Then $\mathbb{N} \vDash \neg\Phi_T$ by axiom soundness. Contradiction with $\mathbb{N} \vDash \Phi_T$. So $T \nvdash \neg\Phi_T$.)

Proof used (i) formulae and proof coding, (ii) diagonal argument.

Remarks

- Φ_T is explicitly constructed.
- Instead of requiring axioms true in $\mathbb{N}$, the weaker requirement of ω-consistency [Gödel], or consistency [Rosser] suffices.
- Similarly, no axiomatisation of analysis or set theory
- Challenges Hilbert's program — with true but not provable statements!

Gödel's Second Theorem (1931)

THEOREM: A consistency proof for any axioms T containing PA cannot be carried out by methods available within T.

PROOF SKETCH

- Gödel's first incompleteness theorem (e.g. Rosser's refinement) shows constructively
  $$T \text{ consistent} \Rightarrow T \nvdash \Phi_T.$$
  ($\Phi_T \equiv \neg Prov_T(\ulcorner \Phi_T \urcorner)$ which essentially says there is no proof of the sentence (i.e. Φ_T) with Gödel number $\ulcorner \Phi_T \urcorner$.)
- Formalising this within PA:
  $$PA \vdash Con_T \to \neg Prov_T(\ulcorner \Phi_T \urcorner), \text{ i.e. } \vdash Con_T \to \Phi_T$$
- Assuming $T \vdash Con_T$, gives $T \vdash \Phi_T$, i.e. $T \vdash \neg Prov_T(\ulcorner \Phi_T \urcorner)$
- Using the explicit Gödel number for the proof $T \vdash \Phi_T$, contradicts $T \vdash \neg Prov_T(\ulcorner \Phi_T \urcorner)$.
- Hence $T \nvdash Con_T$.

References

- [Ham21, chapter 7] is very good.

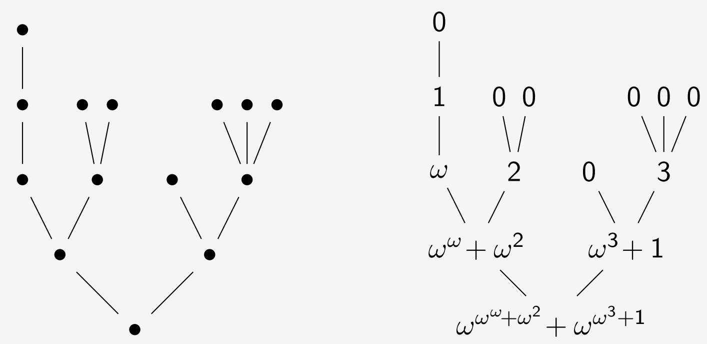
Gödels Second Theorem (continued)

- The true, but not provable from PA, sentence Con_T is of the simple form Π_1.
  Namely $\forall v\, \phi(v)$, where $\phi(v)$ has only bounded quantifiers.
  But Con_T is arguably not a "natural" mathematical sentence/statement.
- "Natural" sentences, true but not provable from PA, ultimately correspond to something codable by the ordinal ϵ_0.
  - Examples include combinatorial Ramsey type theorems, the "hydra" game, and termination of Goodstein processes.
  - Proofs can go via constructing non-standard models $\mathbb{N}^*$ where the relevant result is false, but is true in the standard model $\mathbb{N}$.
  - Alternatively the proofs can go via connections with suitably fast-growing fuctions.
  - Approachable proofs in [Sti10, chap. 6], [KR18, chap 4].

Gentzen (1930's)

- Invents/develops the sequent calculus and natural deduction
  - Logical arguments in tree form, antecedents of a formula are subformulae, initial nodes are axioms
- Proves Cut Elimination (i.e. no modus ponens) at cost of proof sizes exploding hyperexponentially – tetrationally
- Gives consistency proof(s) for PA, assuming:
  - Set of all finite trees, as on next slide, are "well ordered"
  - axioms: just numerical instances (as needed) for functions defined in a primitive recursive manner
- Measures complexity of PA by the ordinal ϵ_0 (next slide)
- His work is the foundation of modern proof theory.
- Kreisel: "Gödel called Gentzen a better logician than himself".

Proof Trees, Natural Deduction, Sequent Calculus



- Nodes of finite trees as on left labelled by ordinals as on right
- top leaves 0 correspond to axioms, e.g. of form $\phi \to \phi$
- node ancestors $\alpha_1 \geq \cdots \geq \alpha_n \Rightarrow$ node is $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$
- Set of all finite trees well-founded order (no ∞ descend. seqs)
- order type of set of all such trees is
  $$\epsilon_0 := \lim\ \omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ \ldots$$

Gödel & Gentzen

- Kurt Gödel (1906–1978): Vienna, IAS Princeton from 1933.
  - From 1936 develops a paranoid fear of being poisoned.
  - Starves himself and dies of malnutrition.
- Gerhard Gentzen (1909–1945): Göttingen, student of Bernays, assistent to Hilbert, Univ of Prague
  - joins Nazi storm troopers, worked on V2 project,
  - but maintains contacts with Bernays, Fraenkel and is denounced by Nazi Teachers Union,
  - arrested by Czech partisans, dies of starvation in prison.

OUTLINE

What is Computable?

- Precise definition of "computable" function $f : \mathbb{N} \to \mathbb{N}$??
  - (Gödel used "primitive recursive" functions, but next comment applies.)
- Can enumerate set of possible instructions, hence list corresponding functions f_n for $n = 1, 2, \ldots$.
  - Define g by $g(n) := f_n(n) + 1$ (diagonalising out), g is computable but not listed.
- Turing: consider algorithmic procedures for computing functions f, which may or may not stop for each input.
  - Turing machines, but think (e.g.) Python programs
  - Gives partially defined functions $f_n : \mathbb{N} \rightharpoonup \mathbb{N}$
  - Diagonalising is not a problem

The Halting Problem

List all Turing machines or all Python programs of following kind:

- n-th machine/program takes inputs $k \in \mathbb{N}$, and outputs $f_n(k) \in \mathbb{N}$ or never halts.

THEOREM
There is no algorithm for deciding if the n-th machine/program with imput n eventually terminates.

Proof Assume there is an algorithm to decide if $f_n(n)$ halts.

- Define

$$g(n) = \begin{cases} f_n(n) + 1 & \text{if } f_n(n) \text{ is defined} \\ 0 & \text{if } f_n(n) \text{ does not halt} \end{cases}$$

- Leads to a contradiction, as g cannot be an f_n. □

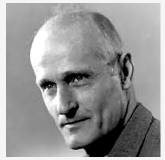
Church-Turing Thesis

- 1936 Turing develops notion of a Turing machine
- 1936 Church develops the λ-calculus, function approach.
- 1936 Kleene the unbounded (least) μ-operator.
- 1936 Post develops Finite Combinatory Processes
- 1936-7 Church, Turing and Kleene separately prove all approaches equivalent.
- *Church's thesis*: Captures the informal notion of computability
- Gödel convinced after he saw Turing's paper, pronounced it *a kind of miracle* that computability had a precise definition.
- Kreisel not convinced (see later, taking into account that Church was a strongly religious Presbyterian)

  Church's thesis has, within logic, a similar function to dogmas and doctrines within the Church. The faithful get excited at the cost of being ridiculous to outsiders.
  [Odi96, Kreisel's Church, pp389–415]

Turing, Church, Kleene, Post






- Turing (1912–1954). Marathon runner, mathematician, theoretical computer science, software and hardware, mathematical biology. Code-breaking UK WWII possibly saved millions of lives. Found "guilty" of private homosexual acts 1952, took hormone treatment ("chemical castration") in lieu of prison. Died of cyanide poisoning, suicide or accidental?
- Church (1903–1995). Princeton, U.C.L.A. Mathematics, philosophy, computer science. Strongly religious.
- Kleene (1909–1994). Student of Church. Univ Wisconsin-Madison. Mountain climber, environmentalist.
- Post (1897-1954). Died from heart attack as a result of electro-shock therapy for depression.

OUTLINE

Tarski

- 1901–1983. Warsaw, Vienna, Berkeley. Logic (model theory, set theory, treatment of truth), philosophy, mathematics.
- Polish Jew who left in 1939 on last boat out before Germany and USSR invaded Poland.
- 1924 Banach-Tarski paradox: a ball can be decomposed into 5 pieces which can be reassembled into two balls, each the same size as the original. (consequence of AC)
- 1942: Founded the Berkeley centre for logic and philosopy of science — successor to Hilbert's Göttingen centre.
- Most U.S. logicians of the 40's and 50's were his students.
- [FF04] for life and work.

Nonstandard Analysis

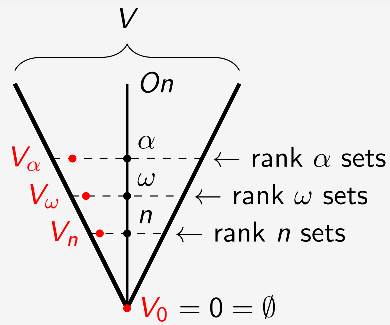
- Abraham Robinson, 1918-74, Berkeley
- Nonstandard Analysis has infinitesimals ϵ, $0 < \epsilon < r$ for all standard real $r > 0$.
- Germ of the idea: from model of $\mathbb{R}$
  - consider $0 < \epsilon < 1/n$ for each $n \in \mathbb{N}$.
  - compactness gives model with infinitesimal ϵ.
- Include first-order set theory for $\mathcal{P}(\mathbb{R})$, $\mathcal{P}^2(\mathbb{R})$, $\mathcal{P}^3(\mathbb{R})$, ...
- Uses relationship between mathematical languages and structures.
- References:
  - Basic calculus Keisler [Kei86], lecturer's manual [Kei22],
  - expository articles and applications on Terry Tao's blog,
  - good development from model theory: [CK12, chap 4.4].

OUTLINE

AC & CH

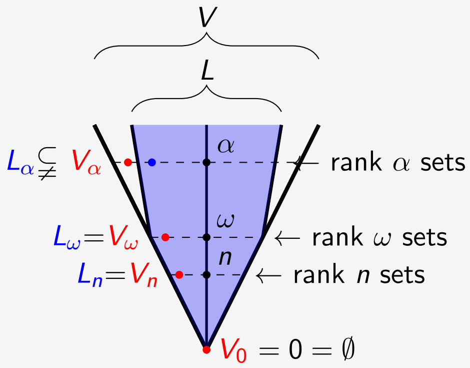
- AC: For any infinite collection of nonempty sets there is a set containing exactly one element from each set.
- Zorn's Lemma: If every linearly ordered in a partially ordered set S has an upper bound, then S has a maximal element.
- Well-ordering Principle: Every set can be well-ordered (i.e. has a linear ordering such that every subset has a least element)
- Theorem: All are equivalent:
  AC $\iff$ Zorn's Lemma $\iff$ Well-ordering principle
- CH: No cardinality between that of $\mathbb{N}$ (or $\mathbb{Q}$) and that of $\mathbb{R}$.
- GCH: No cardinality between that for A and (power set) $\mathcal{P}(A)$.

The Hierarchy V of Sets



- On: collection of ordinals
- Iterate power set along On
- $V_0 = \emptyset$,
  $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,
  $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ (limit λ).
- $\text{rank}(x) = \text{least } \alpha\, (x \subset V_\alpha)$.
- $\text{rank}(\omega) = \text{rank}(V_\omega) = \omega$.
- Tetrational growth:
  - $|V_0| = 0$, $|V_1| = 1$,
  - $|V_2| = 2$, $|V_3| = 2^2 = 4$,
  - $|V_4| = 2^4 = 2^{2^2} = 2 \upuparrows 3 = 16$,
  - $|V_5| = 2^{16} = 2^{2^{2^2}} = 2 \upuparrows 4 = 65,536$,
  - $|V_6| = 2^{65,536} = 2^{2^{2^{2^2}}} = 2 \upuparrows 5 > 10^{19728}$, ...

The Hierarchy L of Constructible Sets



- Iterate $\mathcal{P}^{\text{def}}$ along On
- $L_0 = \emptyset$,
  $L_{\alpha+1} = \mathcal{P}^{\text{def}}(L_\alpha)$,
  $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ (limit λ).

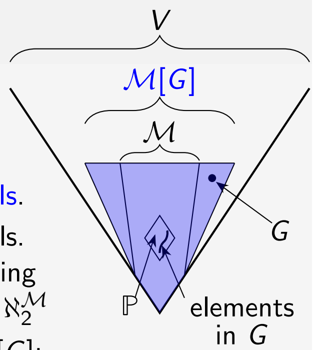
$\mathcal{P}^{\text{def}}(L_\alpha)$ is the set of $u \subset L_\alpha$ which are definable over L_α.

A set $u \subset L_\alpha$ is definable over L_α if $u = \{x \in L_\alpha \mid \phi(x, y_1, \ldots, y_n)\}$ for some first order formula with parameters $y_1, \ldots y_n \in L_\alpha$.

Independence of AC & GCH

- ZF axioms explicitly characterise the sets asserted to exist.
  AC & GCH do not.
- Do AC & GCH follow from ZF axioms?
- 1939: Gödel shows if $V \models ZF$ thens $L \models ZF + AC + GCH$.
  - So ZF consistent implies ZF + AC + GCH consistent.
  - Equivalently, $ZF \not\vdash \neg AC$, $ZF \not\vdash \neg GCH$.
- In retrospect, Gödel's result not surprising. Key ideas:
  - $L \models ZF$: $\mathcal{P}^{\text{def}}$ in L emulates $\mathcal{P}$ in V.
  - $L \models AC$: Construction induces a well-ordering of sets in L.
  - $L \models GCH$: $L_\omega = V_\omega$ countable, hence so is $L_{\omega+1}$, $L_{\omega+2}$, ...
    $\therefore$ cardinality of $L_{\aleph_1}$ is $\aleph_1$.
    $\therefore L \models 2^{\aleph_0} = \aleph_1$.
- 1963: Cohen invents "forcing" to construct models of ZF with additional "generic" sets and in which AC and GCH are false.
  - So Gödel + Cohen $\Rightarrow$ AC, GCH independent of ZF.

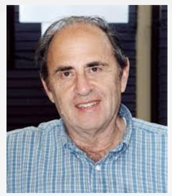
Generic Extension



"METHOD" OF FORCING

- Begin with countable ZFC model $\mathcal{M}$.
- "Fatten" $\mathcal{M}$ to model $\mathcal{M}[G]$, *same* ordinals.
  - very different from nonstandard models.
  - $\mathcal{M}[G]$ has new *generic* set G, containing (e.g.) one-one map between $\mathbb{R}^{\mathcal{M}}$ and $\aleph_2^{\mathcal{M}}$
- Two equivalent approaches to building $\mathcal{M}[G]$:
  - Cohen's original method of "forcing":
    - Start with p.o. $\mathbb{P}$ on finite approxs to intended G.
    - Build G externally to $\mathcal{M}$, using ordinal recursion and countable chain condition.
  - Models $\mathcal{M}^B$ with truth values in Boolean algebra B.
    - Build $\mathcal{M}$ by $\in$-recursion, showing ZF axioms have truth value $\mathbb{1}$.
    - Factor out by an appropriate ultrafilter.

Paul Cohen 1934–2007



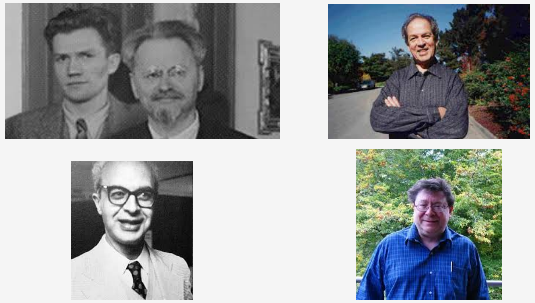
- Stanford from 1961
- 1964 Bôcher prize in analysis
- 1966 Fields Medal
- Ability to prove any major math result "on the spot".
- Would challenge post docs & faculty to explain their most outstanding problem, and then show them how to solve it.

Forcing References

- [Cho08] is a nice introduction with references, including to other applications in topology, topos theory, modal logic, arithmetic, proof theory and computational complexity.
- [Eas08], [Wol05, chap 6.3], [Dža20] are readable introductions.
- For well-written detail see [Coh08], [Kun13, part IV], [Jec03, chap 14], and [Mat] for a forcing overview.
- The recent book [Hal17] looks particularly good.

OUTLINE

Proof Theory at Stanford 1955 and Beyond



van Heijenoort (&Trotsky), Feferman, Kreisel, Friedman

- Centre for proof theory of subsystems of arithmetic & analysis, extending Gentzen's ideas.
- van Heijenoort writes *From Frege to Gödel: A Source Book in Mathematical Logic*.
- Feferman: proof-theoretic strength measured via ordinals.
- Kreisel: unwinding program.
- Friedman: reverse mathematics, subsystems of second order arithmetic. [Eas24]

Proof Theory Stanford 1955 and Beyond

- Jean van Heijenoort, 1912–1986. Secretary, bodyguard to Trotsky 1932–39, Trotsky murdered 1940. PhD NYU 1949. Academic Brandeis, Stanford. Curator Trotsky & Gödel papers. Murdered by 4th wife Mexico. [Fef93].
- Solomon Feferman,1928–2016. Central Stanford figure for 60 years. Etchemendy [Stanford Provost]: "Sol was probably the sweetest man in the world, He had a heart of gold. He and Anita made Stanford ... a welcoming place for a hopelessly insecure graduate student and his wife." (memorial service)
- Georg Kreisel, 1923–2015. Iris Murdoch recorded his brilliance, wit and strangeness, amoralism, cruelty, ambiguous vanity and obscenity [Con01]. Mentor to Feferman and Friedman. Relationship with Feferman ruptured 1972, with Friedman in 1973, with Cohen in 1963. [JS17].
- Harvey Friedman 1948 —. Prodigy. Failed high school, failed MIT undergrad. MIT PhD and Stanford Asst Prof at 18, tenure 20. S.U.N.Y, Ohio State, joint chairs in mathematics, philosophy, computer science, music.

Final Remarks

- Useful to distinguish between
  - the philosophy/foundations of mathematics (proof theory, consistency properties, definitions of truth etc.)
  - the philosophy/foundations of mathematical practice (as practised by most pure mathematicians), see [Har15]
- Other approaches to foundations other than set theory, include category theory, type theory, homotopy type theory. For discussion and further references, see [CKS19].
  - Other approaches should be able to develop, and so essentially include, set theory. Same foundational issues treated in set theory will still be present.
- Many non-classical logics: boolean, modal, tense, conditional, intuitionist, many-valued, paraconsistent, relevant, and fuzzy logics, see [Pri08].
- Current activity on the proof assistant / theorem prover Lean and related material, see [Avi].

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[Coh06] Paul J. Cohen, *Lecture at Gödel Centennial, Vienna*, 2006. https://www.youtube.com/watch?v=VBFLWk7k1Zo. (Cohen discusses the emotional aspects of his discovery, and his interactions with Gödel).

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[Fef93] Anita Feferman, *From Trotsky to Gödel: The Life of Jean Van Heijenoort*, AK Peters/CRC Press, 1993. (A rather extraordinary life. Appendix by Solomon Feferman. The Fefermans knew Van Heijenoort professionally and socially for many years.)

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[HA50] David Hilbert and Wilhelm Ackermann, *Principles of Mathematical Logic*, 2nd ed., translated by Lewis M. Hammond and George G. Luce, Chelsea Publishing Company, New York, 1950. (Translation of the 1938 second edition of *Grundzüge der theoretischen Logik*).

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[HB17] David Hilbert and Paul Bernays, *Foundations of Mathematics*, translated by Claus-Peter Wirth, 2017. (Partial translation of Grundlagen der Mathematik, 1934 & 1968 vol 1, available via http://wirth.bplaced.net/p/hilbertbernays/demos.html. An on-going scholarly and detailed translation of perhaps Hilbert's most important work, with extensive commentary. Vol 2 not yet translated.)

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[Kei22] ______, *Foundations of Infinitesimal Calculus*, 2022. https://people.math.wisc.edu/~hkeisler/foundations.pdf (Lecturer's manual to accompany the author's first year calculus text. Final chapter is a friendly but rigorous introduction to non-standard analysis.)

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Technical Notes

- At top level, VSCode; then Github for syncing a remote repository and 2 local repositories.
- Slides written using LaTeX Workshop as a VSCode extension:
  - `\documentclass[handout]beamer`
  - and a mixture of various other packages
- Two other VSCode extensions were helpful:
  - GitHub Copilot for occasional phrasing suggestions,
  - and GitHub Copilot Chat for compiling references and for help with LaTeX and various packages such as Tikz.
- Presentation by mirroring pdf slides from Goodnotes app on an iPad to main screens.
  - Goodnotes has useful presentation tools, in particular the apple pencil works as a "laser pointer".
- Photos from Wikipedia, Public Domain.