Leaving Certificate Examination, 1962

Mathematics II

HONOURS PAPER

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Time allowed — Three hours

Candidates may submit answers to *eight* questions only.

The questions are of equal value.

(*John Hutchinson – Solutions & Comments;*
March 17, 2021)

Question 1

(i) Prove that

$$\left(1+\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right)^n=-2^n\cos^n\frac{\pi}{n}.$$

(ii) Prove that, if n is any integer, positive or negative,

$$\left(-1+i\sqrt{3}\right)^n+\left(-1-i\sqrt{3}\right)^n$$

has either the value 2^{n+1} or the value -2^n.

Solution.

1. (i)

$$1+\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}=1+e^{2\pi i/n}=e^{\pi i/n}\left(e^{-\pi i/n}+e^{\pi i/n}\right)=2\cos\frac{\pi i}{n}\;e^{\pi i/n}.$$

Hence

$$\left(1+\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right)^n=2^n\cos^n\frac{\pi}{n}\;e^{\pi i}=-2^n\cos^n\frac{\pi}{n}.$$

(ii)

$$-1+i\sqrt{3}=2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=2\,e^{2\pi i/3}.$$

Similarly,

$$-1-i\sqrt{3}=2\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=2\left(\cos-\frac{2\pi}{3}+i\sin-\frac{2\pi}{3}\right)=2\,e^{-2\pi i/3}.$$

Taking the nth power of each side and adding gives

$$\left(-1+i\sqrt{3}\right)^n+\left(-1-i\sqrt{3}\right)^n=2^n\left(e^{2n\pi i/3}+e^{-2n\pi i/3}\right)=2^{n+1}\cos\frac{2n\pi}{3}. \tag{1}$$

If $n=3k$ for some integer k, then

$$\cos 2n\pi/3=\cos 2k\pi=\cos 0=1. \tag{2}$$

If $n=3k\pm1$ then

$$\cos 2n\pi/3=\cos(2k\pi\pm 2\pi/3)=\cos\pm 2\pi/3=-1/2. \tag{3}$$

The required result follows from (1), (2) and (3). □

Tips & Tricks The idea in both parts is to write $a+ib$ in the form $re^{i\theta}$ before taking powers.

Question 2

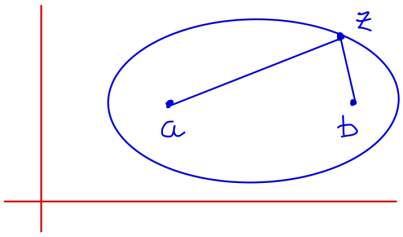
2. (i) Draw sketches to show the loci represented on the Argand diagram by
(a) $|z-a|=k$,
(b) $\arg(z-a)=\alpha$,
(c) $|z-a|+|z-b|=c$,
where a, b are complex constants, k, c, α are real constants, and $c > |a-b|$.

(ii) Prove that, if $z \neq 0$,
(a) $u = z + \dfrac{|z|^2}{z}$ is always real,
(b) $v = \dfrac{|z| - iz}{|z| + iz}$ is always pure imaginary, provided the real part of z is not zero, and that, if $\arg z = \theta$, then $v = -i(\sec\theta + \tan\theta)$.

Solution.

2. (i)

(a) The first sketch is the circle represented by $|z-a|=k$.
(b) The second sketch is the line segment given by $\arg(z-a)=\alpha$.



(c) The third sketch is the ellipse given by $|z-a|+|z-b|=c$, where the line segments connecting z to a and b have lengths $|z-a|$ and $|z-b|$ respectively, and sum to c.

(ii) Let $z = re^{i\theta}$ in polar form. By assumption, $r \neq 0$.

(a) We have

$$u = z + \frac{|z|^2}{z} = re^{i\theta} + \frac{r^2}{re^{i\theta}} = r\left(e^{i\theta} + e^{-i\theta}\right) = 2r\cos\theta,$$

which in particular is real.

(b) By assumption, z has real part not zero, or equivalently z is not pure imaginary, or equivalently in polar form $\theta \neq \frac{\pi}{2}, \frac{3\pi}{2}$. Then

$$\begin{aligned} v = \frac{|z| - iz}{|z| + iz} &= \frac{r - ire^{i\theta}}{r + ire^{i\theta}} = \frac{1 - ie^{i\theta}}{1 + ie^{i\theta}} = \frac{1 - ie^{i\theta}}{1 + ie^{i\theta}} \times \frac{1 - ie^{-i\theta}}{1 - ie^{-i\theta}} = \frac{1 - ie^{-i\theta} - ie^{i\theta} - 1}{1 - ie^{-i\theta} + ie^{i\theta} + -1} \\ &= \frac{-2i\cos\theta}{2 - 2\sin\theta} = -i\frac{\cos\theta}{1 - \sin\theta} = -i\frac{\cos\theta}{1 - \sin\theta} \times \frac{1 + \sin\theta}{1 + \sin\theta} \\ &= -i\frac{\cos\theta(1 + \sin\theta)}{\cos^2\theta} = -i\frac{1 + \sin\theta}{\cos\theta} = -i(\sec\theta + \tan\theta). \end{aligned}$$

The assumptions on z were used to ensure the denominator in the definition of v is non-zero, and the denominators $1 - ie^{-i\theta}$ and $1 + \sin\theta$ introduced in the first and second lines above are also non-zero. □

Tips & Tricks In part (ii) first write z in the form $re^{i\theta}$. Then part (ii)(a) is easy.

For part (ii)(b), in the first line for the calculation of v, the term $(1 - ie^{-i\theta})$ in the introduced factor $1 = (1 - ie^{-i\theta})/(1 - ie^{-i\theta})$ is just the complex conjugate of $1 + ie^{i\theta}$. In this way the new denominator of v is real. (So this is a good move)

In the second line of the calculation the term $1 + \sin\theta$, in the new factor $1 = (1 + \sin\theta)/(1 + \sin\theta)$, has the property that when multiplied by $1 - \sin\theta$ the result is $\cos^2\theta$ (which is the second good move).

Question 3

Part 1 F is the point $(ae, 0)$ and d is the line $x = a/e$ $(e > 1)$. M is the foot of the perpendicular from a variable point P to d, and P moves so that

$$FP^2 = e^2 PM^2.$$

Find the equation of the locus.

Draw a sketch showing clearly the principal axes of the curve, the foci, the directrices and the asymptotes, marking on each of them its equation or coordinates.

Express e in terms of α, the angle between the asymptotes.

Part 2 A line, parallel to one asymptote, is drawn through a focus F of a hyperbola. It meets the hyperbola in H, the directrix corresponding to F in D, the other asymptote in K and the conjugate axis in R. Prove that

$$FH = HD,$$
$$FK = KR.$$

Solution.

Part 1

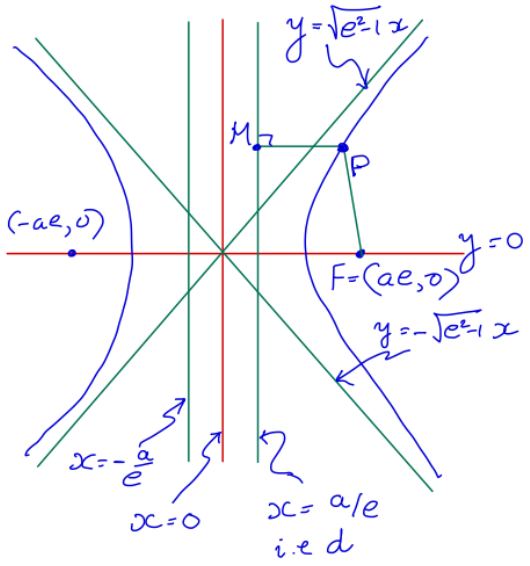


Figure 1: Hyperbola with focus F and directrix d, traced out by P moving so $FP^2 = e^2 PM^2$.

See Figure 1. Let P be the point (x, y). Then $M = (a/e, y)$. Since $FP^2 = e^2 PM^2$, it follows that

$$(x - ae)^2 + y^2 = e^2(x - a/e)^2 = (ex - a)^2$$
$$\therefore \quad a^2(e^2 - 1) = x^2(e^2 - 1) - y^2$$
$$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1. \tag{4}$$

The last equation is the equation of the locus. It is in the standard form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ for an hyperbola, with $b = a\sqrt{e^2 - 1}$.

It follows from standard formulae for an ellipse that

- the principal axes are $y = 0$ and $x = 0$,
- the foci are $\pm\left(\left(a^2+b^2\right)^{1/2}, 0\right) = \pm(ae, 0)$,
- the directices are $x = \pm\dfrac{a^2}{\sqrt{a^2+b^2}}$, i.e. $x = \pm a/e$ (i.e. the line d and its mirror image in the y-axis),
- and the asymptotes are $y = \pm\dfrac{b}{a}x$, i.e. $y = \pm\sqrt{e^2-1}\,x$.

If α is the angle between the asymptotes then $\tan(\alpha/2) = \sqrt{e^2-1}$ and so $\alpha = 2\arctan\sqrt{e^2-1}$. We can also simplify this as follows:

$$\begin{aligned} \tan^2\frac{\alpha}{2} &= e^2 - 1 \\ \therefore \quad \sin^2\frac{\alpha}{2} &= e^2\cos^2\frac{\alpha}{2} - \cos^2\frac{\alpha}{2} \\ \therefore \quad 1 &= e^2\cos^2\frac{\alpha}{2} \\ \therefore \quad \cos\frac{\alpha}{2} &= e^{-1} \\ \therefore \quad \alpha &= 2\arccos\left(e^{-1}\right). \end{aligned}$$

Part 2

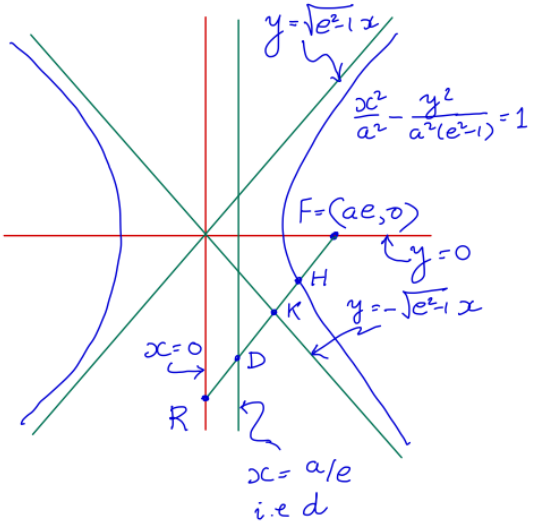


Figure 2: $FH = HD$, $FK = KR$.

See Figure 2. We can take the hyperbola to be the same as that in Part 1, since the most general hyperbola can be represented in the standard form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and (4) is in this form if we take $e = \sqrt{1+\frac{b^2}{a^2}}$.

The equation of the line through F and parallel to the asymptote $y = \sqrt{e^2-1}\,x$ is

$$y = \sqrt{e^2-1}\,(x - ae)$$

since this has the required slope and passes through F. (If we took the second asymptote the following argument would be similar. But it is also clear that it is sufficient to just do one case, as the other in essence follows geometrically be reflecting everything in the x-axis.)

In order to prove $FH = HD$ and $FK = KR$, since these are parallel segments, it is sufficient to prove that the corresponding projections onto the x-axis are equal. We write FH_x etc. for these projections, and similarly write F_x, H_x etc. for the x-coordinates of F, H etc.

Then

- $F_x = ae$.
- At H, $y = \sqrt{e^2-1}\,(x-ae)$ and $x^2 - y^2/(e^2-1) = a^2$. Hence $x^2 - (x-ae)^2 = a^2$, therefore $2xae = a^2(1+e^2)$ and so $x = a/(2e) + ae/2$. That is

$$H_x = \frac{a}{2e} + \frac{ae}{2}.$$

- At K, $y = \sqrt{e^2-1}(x-ae)$ and $y = -\sqrt{e^2-1}\,x$, so $x = ae/2$. That is

$$K_x = \frac{ae}{2}.$$

- Clearly,

$$D_x = \frac{a}{e}, \qquad R_x = 0.$$

It follows that

$$\begin{aligned} FH_x &= F_x - H_x = \frac{a}{2e} - \frac{ae}{2}, \\ HD_x &= H_x - D_x = \frac{a}{2e} - \frac{ae}{2}, \\ FK_x &= F_x - K_x = \frac{ae}{2}, \\ KR_x &= K_x - R_x = \frac{ae}{2}. \end{aligned}$$

This proves the required equalities. □

Question 4

Express $x^n - 1$ as a product of real linear or quadratic factors, distinguishing the cases n odd and n even.

Thence or otherwise prove that

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\left(\frac{n-1}{2n}\pi\right) = \sqrt{\frac{n}{2^{n-1}}}$$

when n is odd, and find the corresponding result when n is even.

Proof.

First assume n is odd. Then the solutions of $x^n = 1$ can be written in the form[1]

$$x = 1, \qquad x = e^{\pm\frac{2\pi i}{n}j} \quad \text{for } j = 1, \ldots, \frac{n-1}{2}.$$

Hence

$$\begin{aligned} x^n - 1 &= (x-1)\prod_{j=1}^{\frac{n-1}{2}}\left(x - e^{\frac{2\pi i}{n}j}\right)\left(x - e^{-\frac{2\pi i}{n}j}\right) \\ &= (x-1)\prod_{j=1}^{\frac{n-1}{2}}\left(x^2 - 2x\cos\left(\frac{2\pi j}{n}\right) + 1\right). \end{aligned} \tag{5}$$

This is a product of real linear and quadratic factors, as required.

Dividing both sides by $x - 1$ it follows that for $x \neq 1$

$$1 + x + x^2 + \cdots + x^{n-1} = \prod_{j=1}^{\frac{n-1}{2}}\left(x^2 - 2x\cos\left(\frac{2\pi j}{n}\right) + 1\right). \tag{6}$$

Since both sides are polynomials in x the limit as $x \to 1$ of both sides are equal, and so

$$n = \prod_{j=1}^{\frac{n-1}{2}}\left(2 - 2\cos\left(\frac{2\pi j}{n}\right)\right) = \prod_{j=1}^{\frac{n-1}{2}} 4\sin^2\left(\frac{\pi j}{n}\right) = 2^{n-1}\prod_{j=1}^{\frac{n-1}{2}}\sin^2\left(\frac{\pi j}{n}\right),$$

using $\cos(2\theta) = 1 - 2\sin^2\theta$. Taking the square root of each side and noting each of the sine terms is positive,

$$\prod_{j=1}^{\frac{n-1}{2}}\sin\left(\frac{\pi j}{n}\right) = \sqrt{\frac{n}{2^{n-1}}}, \tag{7}$$

as required.

If n is even, then the solutions of $x^n = 1$ can be written in the form

$$x = \pm 1, \qquad x = e^{\pm\frac{2\pi i}{n}j} \quad \text{for } j = 1, \ldots, \frac{n}{2} - 1.$$

Hence

$$\begin{aligned} x^n - 1 &= (x-1)(x+1)\prod_{j=1}^{\frac{n}{2}-1}\left(x - e^{\frac{2\pi i}{n}j}\right)\left(x - e^{-\frac{2\pi i}{n}j}\right) \\ &= (x-1)(x+1)\prod_{j=1}^{\frac{n}{2}-1}\left(x^2 - 2x\cos\left(\frac{2\pi j}{n}\right) + 1\right). \end{aligned} \tag{8}$$

[1] The solutions are uniformly distributed around the unit circle in the complex plane. They are written here in complex conjugate pairs, apart from the solution $x = 1$.

This is again a product of real linear and quadratic factors, as required.

Dividing both sides by $x-1$ it follows that for $x \neq 1$

$$1 + x + x^2 + \cdots + x^{n-1} = (x+1) \prod_{j=1}^{\frac{n}{2}-1} \left(x^2 - 2x \cos\left(\frac{2\pi j}{n}\right) + 1 \right).$$

Once again taking the limit as $x \to 1$,

$$n = 2 \prod_{j=1}^{\frac{n}{2}=1} \left(2 - 2\cos\left(\frac{2\pi j}{n}\right)\right) = 2 \prod_{j=1}^{\frac{n}{2}-1} 4\sin^2\left(\frac{\pi j}{n}\right) = 2 \times 2^{n-2} \prod_{j=1}^{\frac{n}{2}-1} \sin^2\left(\frac{\pi j}{n}\right) = 2^{n-1} \prod_{j=1}^{\frac{n}{2}-1} \sin^2\left(\frac{\pi j}{n}\right).$$

Again taking the square root of both sides,

$$\prod_{j=1}^{\frac{n}{2}-1} \sin\left(\frac{\pi j}{n}\right) = \sqrt{\frac{n}{2^{n-1}}}, \tag{9}$$

as required. □

Tips & Tricks Obtaining the factorisation in (5) is not that difficult, once you know how to pair conjugate roots to get real quadratic factors. More precisely, if $x = w$ and $x = \overline{w}$ are both solutions of $x^n = 1$, the $x - w$ and $x - \overline{w}$ are both factors and hence so is

$$(x-w)(x-\overline{w}) = x^2 - 2x\,\mathrm{Re}(w) + |w|^2,$$

which is real quadratic and where Re means "the real part of".

But the other problem in getting to (7) from (5) is that the angles in (5) have increments $2\pi/n$, whereas in (7) the incrments are π/n. This leads one to considering the formula $\cos(2\theta) = 1 - 2\sin^2\theta$, which indicates the approach to take.

The final twist is that one really needs $x = 1$ in (6), whereas it was derived under the assumption $x \neq 1$. The more general fact that has been used is that if $f(x) = (x-a)\,g(x)$ for $x = a$ and all x near a, then $f'(a) = g(a)$. (Assuming g is continuous at a. *Exercise.*)

Question 5

Draw a sketch to show the relation between the point $(a\cos\phi, b\sin\phi)$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the corresponding point on the auxiliary circle.[2]

Write down the equation of the circle on which the points (x_1, y_1), (x_2, y_2) are the extemities of a diameter.

P is a variable point on an ellipse of which F is one focus. Prove the circle on PF as diameter touches the auxiliary circle of the ellipse.

Solution.

Without loss of generality we take $a > b$.

1.

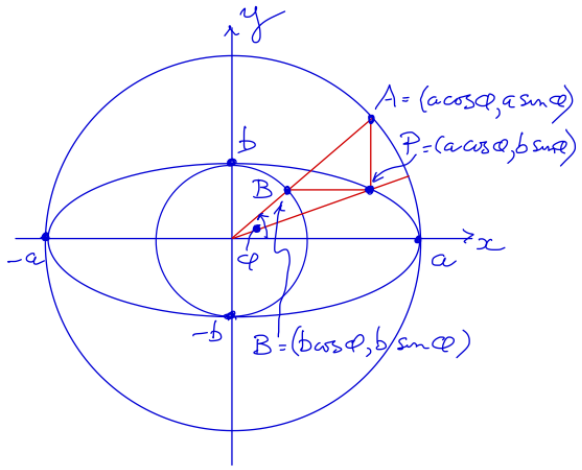


Figure 3: The larger circle is the auxiliary circle for the ellipse.

See Figure 3. The question is a little ambiguous, but the point corresponding to P on the auxiliary circle (outer) circle O is usually taken to be $A = (a\cos\phi, a\sin\phi)$. (The point corresponding to P on the inner circle is $B = (b\cos\phi, b\sin\phi)$.)

2. The equation of the circle on which the points (x_1, y_1), (x_2, y_2) are the extemities of a diameter is

$$(x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$$

which simplifies to

$$x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + (x_1x_2 + y_1y_2) = 0. \tag{10}$$

3 See Figure 4. The focus F has coordinates $(\pm c, 0)$ where

$$c > 0, \quad c^2 = a^2 - b^2. \tag{11}$$

Without loss of generality because of symmetry, we take $F = (c, 0)$.

The point P on the ellipse has coordinates $(a\cos\phi, b\sin\phi)$ for some $\phi \in [0, 2\pi)$. Fix ϕ. From (10), the equation of the circle C on PF as diameter is

$$x^2 + y^2 - x(a\cos\phi + c) - y\,b\sin\phi + a\,c\cos\phi = 0. \tag{12}$$

[2]The auxiliary circle for an ellipse is the circle whose diameter is the major axis of the ellipse.

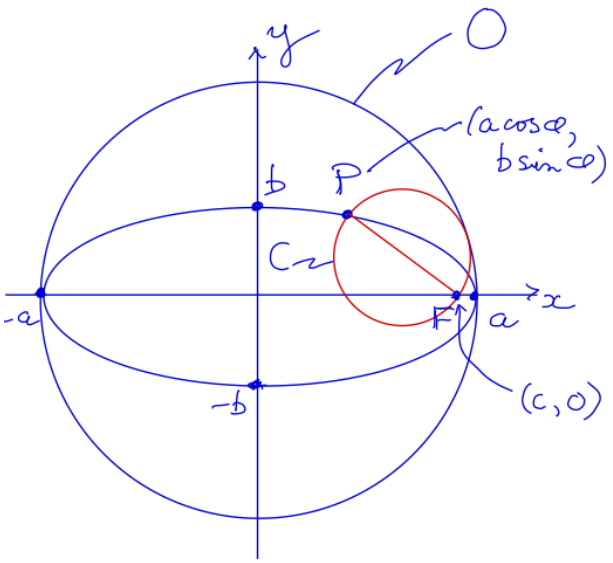


Figure 4: Ellipse with auxiliary circle O. The circle C with diameter PF touches O.

In order to prove the circle C on PF as diameter touches the auxiliary circle O of the ellipse, we will show that every point $(x, y) \in O$ satisfies

$$x^2 + y^2 - x(a\cos\phi + c) - y\,b\sin\phi + a\,c\cos\phi \geq 0, \tag{13}$$

and that some point $(x, y) \in O$ satisfies (12).

Take an arbitrary point $(x, y) = (a\cos\theta, a\sin\theta) \in O$, and substituting into the left side of (13) we want to show

$$a\big(a - \cos\theta(a\cos\phi + c) - b\sin\theta\sin\phi + c\cos\phi\big) \geq 0 \tag{14}$$

for all θ, and show that equality holds for some θ.

Using the trigonometric identities

$$t = \tan\frac{\theta}{2}, \quad \sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}, \quad \tan\theta = \frac{2t}{1-t^2}, \tag{15}$$

we want to show for all t that

$$a - a\frac{1-t^2}{1+t^2}\cos\phi - b\frac{2t}{1+t^2}\sin\phi + c\left(\cos\phi - \frac{1-t^2}{1+t^2}\right) \geq 0, \tag{16}$$

and that equality holds for some t.

Multiplying through by the positive term $1 + t^2$ we obtain the following quadratic in t:

$$\begin{aligned} &a(1+t^2) - a(1-t^2)\cos\phi - 2bt\sin\phi + c(1+t^2)\cos\phi - c(1-t^2) \\ &= t^2(a + a\cos\phi + c\cos\phi + c) - 2bt\sin\phi + (a - a\cos\phi + c\cos\phi - c) \\ &= t^2(a+c)(1+\cos\phi) - 2bt\sin\phi + (a-c)(1-\cos\phi). \end{aligned} \tag{17}$$

Assuming initially that $\phi \neq \pi$, it follows that the coefficient of t^2 is > 0. The discriminant of the quadratic is

$$4\left(b^2\sin^2\phi - (a^2 - c^2)(1 - \cos^2\phi)\right) = 4(b^2 - a^2 + c^2)\sin^2\phi = 0, \tag{18}$$

using (11).

Since the quadratic (17) has zero discriminant and positive leading term, it is always ≥ 0, and takes the minimum value 0. Moreover, this minimum value occurs at

$$t = \frac{2b\sin\phi}{2(a+c)(1+\cos\phi)} = \frac{b\sin\phi}{(a+c)(1+\cos\phi)}. \tag{19}$$

This answers the question if $\phi \neq \pi$, since it follows that the circle C lies inside the auxiliary circle O and meets the auxiliary circle when t is as in (19). Moreover, C and the auxiliarly circle must touch tangentially from geometric considerations. This also follows from the fact that the derivative of the quadratic (17) at the minimum is zero.

Finally, if $\phi = \pi$ then $P = (-\pi, 0)$, and from the previous diagram it is clear in this case that C lies inside the auxiliary circle and the two circles touch at P. □

Tips & Tricks Once we have reached (14), it is reasonable to think about minimising a quadratic, and to guess that one possibility might be to use the half angle identities in (15). Fortunately, with a little care, this does work.

Addendum

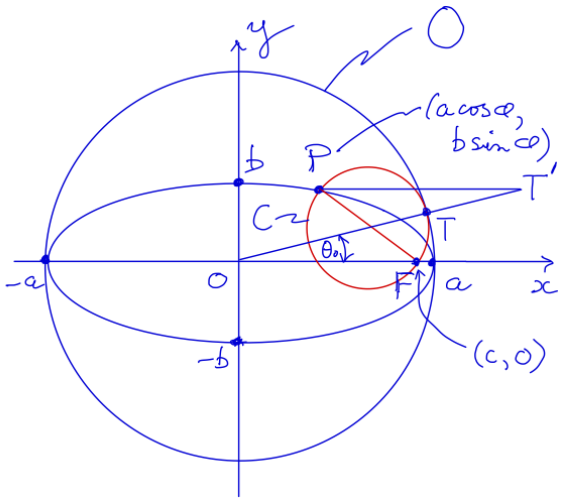


Figure 5: Geometric construction of the touching point T of the red circle and the auxiliary circle. PT' is parallel to and has equal length to OF.

From (15) and (19), the value of θ_0 for the point $T = (a\cos\theta_0, a\sin\theta_0)$ at which C touches the auxiliary circle, satisfies

$$\tan\theta_0 = \frac{2t}{1-t^2}, \tag{20}$$

where t is as in (19).

We calculate

$$\begin{aligned} 1 - t^2 &= 1 - \frac{b^2\sin^2\phi}{(a+c)^2(1+\cos\phi)^2} \\ &= \frac{(a+c)^2(1+\cos\phi)^2 - (a^2-c^2)(1-\cos^2\phi)}{(a+c)^2(1+\cos\phi)^2} \qquad \text{(from (11))} \\ &= \frac{(a+c)(1+\cos\phi) - (a-c)(1-\cos\phi)}{(a+c)(1+\cos\phi)} \\ &= \frac{2(a\cos\phi + c)}{(a+c)(1+\cos\phi)}. \end{aligned}$$

From this, (19) and (20) it follows that

$$\tan\theta_0 = \frac{b\sin\phi}{a\cos\phi + c}, \quad \theta_0 = \arctan\frac{b\sin\phi}{a\cos\phi + c}.$$

Looking at Figure 5, it follows that the point T can be found as follows. Draw a line parallel to the x-axis through P and move taway from the y-axis by a distance equal to the focal length c. Call this point T' and draw a straight line from the origin through T'. Then T is the intersection point of this line with the auxiliary circle. (*Why?*)

Question 6

Prove that the feet of the four normals to the ellipse, $E = 0$, where $E \equiv x^2/a^2 + y^2/b^2 - 1$, from the point (x_1, y_1) are the intersections of $E = 0$ with the rectangular hyperbola $H_1 = 0$, where

$$H_1 \equiv \left(\frac{1}{b^2} - \frac{1}{a^2}\right) xy + \frac{y_1}{a^2}\, x - \frac{x_1}{b^2}\, y.$$

Give a reason why, if $\lambda H_1 + E = 0$ represents a line pair[3], then this line pair is one of the three which join pairs of the feet of the four normals.

The join of the feet of two of these four normals is the line

$$\frac{lx}{a} + \frac{my}{b} + 1 = 0;$$

prove that the join of the other two feet is

$$\frac{x}{la}m + \frac{y}{mb} - 1 = 0.$$

Find the values of x_1, y_1 in terms of l, m.

Solution.

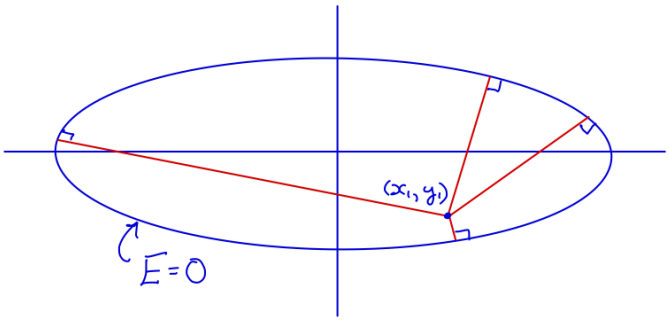


Figure 6: Ellipse and the four normals from a point (x_1, y_1).

1 The normal to $E = 0$ at (x, y), which is the direction of maximum increase in $E = E(x, y)$ at (x, y), is given by

$$\left(\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y}\right) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}\right) = 2\left(\frac{x}{a^2}, \frac{y}{b^2}\right).$$

So $P = (x, y)$ is the foot of a normal from (x_1, y_1) to $E = 0$ if and only if

(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$, and

(ii) $\dfrac{x - x_1}{x/a^2} = \dfrac{y - y_1}{y/b^2}$.

We can rewrite (ii) as

$$\left(\frac{1}{b^2} - \frac{1}{a^2}\right) xy + \frac{y_1}{a^2}\, x - \frac{x_1}{b^2}\, y = 0.$$

[3]This means that $\lambda H_1 + E$ can be written as a product of two linear factors. Note that in this case, if both $H_1 = 0$ and $E = 0$ then at least one of the two linear factors equals zero.

Hence the feet of the four normals to $E = 0$ from (x_1, y_1) are the intersections of $E = 0$ with $H_1 = 0$.

2 Suppose $\lambda H_1 + E = 0$ represents the line pair $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. This means that for some a_1, b_1, c_1 and a_2, b_2, c_2, and for all x, y,

$$\lambda H_1 + E \equiv (a_1x + b_1y + c_1)(a_2x + b_2y + c_2). \tag{21}$$

At each of the four points $P = P(x, y)$ which are the feet of the four normals, one has $E = 0$ and $H_1 = 0$ and so either $a_1x + b_1y + c_1 = 0$ or $a_2x + b_2y + c_2 = 0$. Since no three points on $E = 0$ are in a straight line, two of the feet must satisfy $a_1x + b_1y + c_1 = 0$ and the other two satisfy $a_2x + b_2y + c_2 = 0$.

In other words, the line pair is one of the three which join pairs of the feet of the four normals.

3 It is given that the join of the feet of two of these four normals is the line

$$\frac{lx}{a} + \frac{my}{b} + 1 = 0.$$

(Note that any line can be written in this form.)

It follows from (21) that the join of the other two feet can be written in the form $\alpha x + \beta y + \gamma = 0$, where

$$\lambda\left(\left(\frac{1}{b^2} - \frac{1}{a^2}\right)xy + \frac{y_1}{a^2}\,x - \frac{x_1}{b^2}\,y\right) + \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = \left(\frac{lx}{a} + \frac{my}{b} + 1\right)(\alpha x + \beta y + \gamma) \tag{22}$$

for all x, y.

Equating the coefficients of x^2, y^2 and the constant terms, gives rspectively,

$$\alpha = \frac{1}{la}, \quad \beta = \frac{1}{mb}, \quad \gamma = -1. \tag{23}$$

That is, the join of the other two feet is

$$\frac{x}{la} + \frac{y}{mb} - 1 = 0. \tag{24}$$

4 From (22), using (23) and equating the coefficients of xy,

$$\begin{aligned} \lambda\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = \frac{l\beta}{a} + \frac{m\alpha}{b} = \frac{l}{abm} + \frac{m}{abl} = \frac{l^2 + m^2}{ablm} \\ \therefore \quad \lambda = \frac{ab(l^2 + m^2)}{(a^2 - b^2)lm}. \end{aligned} \tag{25}$$

Again from (22), equating the coefficients of x and of y and using (23) and (25), gives respectively

$$\begin{aligned} y_1 = \frac{a^2\alpha}{\lambda} = a^2\,\frac{1}{la}\,\frac{(a^2 - b^2)lm}{ab\,(l^2 + m^2)} = \frac{(a^2 - b^2)m}{b\,(l^2 + m^2)}, \\ x_1 = -\frac{b^2\beta}{\lambda} = -b^2\,\frac{1}{mb}\,\frac{(a^2 - b^2)lm}{ab\,(l^2 + m^2)} = -\frac{(a^2 - b^2)l}{a\,(l^2 + m^2)}. \end{aligned} \tag{26}$$

□

Tips & Tricks

1. The first point is that the normal to $E = 0$ points in the direction of maximum increase ot the quantity E. Ths is a general and important fact.
2. The idea of a quadratic in two variables (here $\lambda H_1 + E$ for some λ) being factorisable into two linear factors is interesting and used in the subject of algebraic geometry.
3. Note how (25) and (26) are proved by equating coefficients.

Question 7

Sketch the graph of the function

$$x = \tan y$$

and indicate how a range of principal values of the inverse tangent should be chosen. Denoting the principal value of he inverse tangent by $\tan^{-1} x$, give a formula for the general value of the inverse tangent.

If a sequence is defined by

$$\begin{aligned} u_0 &= 1 \\ u_{n+1} &= \frac{2^{n+1}u_n + 1}{2^{n+1} - u_n} \end{aligned} \tag{27}$$

show that

$$\lim_{n\to\infty} \tan^{-1} u_n = \frac{\pi}{4} + \sum_{1}^{\infty} \tan^{-1} \frac{1}{2^n}. \tag{28}$$

Using the tables compute the value of this limit to two significant figures.

Spoiler The limit (28) is wrong, see (39).

Solution.

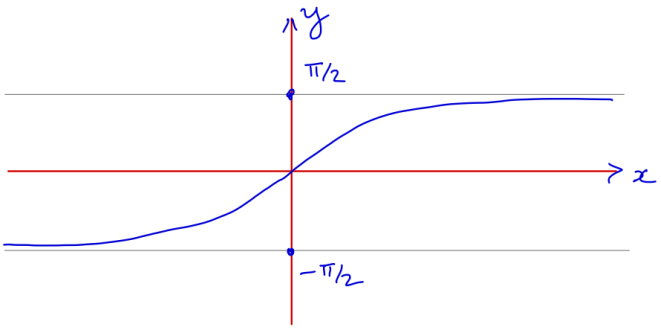


Figure 7: Graph of $y = \tan^{-1} x$ (i.e. of $x = \tan y$ for $y \in (-\pi/2, \pi/2)$)

1 The *principal values* of the inverse function $y = \tan^{-1} x$ are chosen in the open interval $(-\pi/2, \pi/2)$. The *general value* of the inverse tangent of x is $\tan^{-1} x + k\pi$ for k an arbitrary integer.

2 The formula for the tangent of the sum of two angles is

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}, \tag{29}$$

provided $\tan\alpha\tan\beta \neq 1$.

Suppose $u, v \in \mathbb{R}$. Let $\alpha = \tan^{-1} u$ and $\beta = \tan^{-1} v$, in which case $\alpha, \beta \in (-\pi/2, \pi/2)$ and $u = \tan\alpha, v = \tan\beta$. It follows from (29) that

$$\tan(\tan^{-1} u + \tan^{-1} v) = \frac{u + v}{1 - uv}, \tag{30}$$

and so

$$\tan^{-1} \frac{u + v}{1 - uv} = \tan^{-1} u + \tan^{-1} v, \tag{31}$$

provided

$$uv \neq 1, \quad \tan^{-1} u + \tan^{-1} v \in (-\pi/2, \pi/2). \tag{32}$$

In order to apply (31) to (27), write (27) in the form

$$u_{n+1} = \frac{u_n + 2^{-(n+1)}}{1 - u_n 2^{-(n+1)}}.$$

It follows

$$\tan^{-1} u_{n+1} = \tan^{-1} u_n + \tan^{-1} 2^{-(n+1)}, \tag{33}$$

provided

$$u_n 2^{-(n+1)} \neq 1, \quad \tan^{-1} u_n + \tan^{-1} 2^{-(n+1)} \in (-\pi/2, \pi/2). \tag{34}$$

Hence, except for the fact that we have not yet checked the second condition in (34),

$$\begin{aligned}
\tan^{-1} u_1 &= \tan^{-1} u_0 + \tan^{-1} \frac{1}{2} = \frac{\pi}{4} + \tan^{-1} \frac{1}{2} \\
\tan^{-1} u_2 &= \tan^{-1} u_1 + \tan^{-1} \frac{1}{2^2} = \frac{\pi}{4} + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{2^2} \\
\tan^{-1} u_3 &= \tan^{-1} u_2 + \tan^{-1} \frac{1}{2^3} = \frac{\pi}{4} + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{2^2} + \tan^{-1} \frac{1}{2^3} \\
&\vdots \\
\tan^{-1} u_n &= \frac{\pi}{4} + \sum_{k=1}^{n} \tan^{-1} \frac{1}{2^k} \\
&\vdots
\end{aligned} \tag{35}$$

Therefore,

$$\lim_{n\to\infty} \tan^{-1} u_n = \frac{\pi}{4} + \sum_{1}^{\infty} \tan^{-1} \frac{1}{2^n},$$

as required.

It does remain, however, to check that (34) holds for each line in (35). We have, noting $\pi/2 = 1.5707963268\ldots$,

$\tan^{-1} u_0 + \tan^{-1} \frac{1}{2} = \frac{\pi}{4} + \tan^{-1} \frac{1}{2} \approx 1.249 < \pi/2$, so (34) applies and $\tan^{-1} u_1$ is as in (35),

$\tan^{-1} u_1 + \tan^{-1} \frac{1}{2^2} = \frac{\pi}{4} + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{2^2} \approx 1.494 < \pi/2$, so again $\tan^{-1} u_2$ is as in (35),

$\tan^{-1} u_2 + \tan^{-1} \frac{1}{2^3} = \frac{\pi}{4} + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{2^2} + \tan^{-1} \frac{1}{2^3} \approx 1.618 > \pi/2$, so (34) does not apply.

This means we *cannot* apply (34) and (33) in this case.

However, suppose in (30) that $\tan^{-1} u + \tan^{-1} v \in (\pi/2, 3\pi/2)$. In this case it follows from (30) and **1** that

$$\tan^{-1} \frac{u+v}{1-uv} = \tan^{-1} u + \tan^{-1} v - \pi, \tag{36}$$

Hence, correcting (35), one has

$$\tan^{-1} u_3 = \tan^{-1} u_2 + \tan^{-1} \frac{1}{2^3} - \pi = -\frac{3\pi}{4} + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{2^2} + \tan^{-1} \frac{1}{2^3} \approx -1.523. \tag{37}$$

Note that $-1.523 \in (-\pi/2, \pi/2)$.

We can now continue as in (35) and obtain

$$\begin{aligned}
\tan^{-1} u_3 &= \tan^{-1} u_2 + \tan^{-1} \frac{1}{2^3} - \pi = -\frac{3\pi}{4} + \sum_{k=1}^{3} \tan^{-1} \frac{1}{2^k} \\
\tan^{-1} u_4 &= \tan^{-1} u_2 + \tan^{-1} \frac{1}{2^3} - \pi = -\frac{3\pi}{4} + \sum_{k=1}^{4} \tan^{-1} \frac{1}{2^k} \\
&\vdots \\
\tan^{-1} u_n &= -\frac{3\pi}{4} + \sum_{k=1}^{n} \tan^{-1} \frac{1}{2^k} \\
&\vdots
\end{aligned} \tag{38}$$

This time the argument is correct, since for all $n \geq 3$, (and using $\tan^{-1} x < x$ for all $x > 0$ in the second line below),

$$\begin{aligned}
-\frac{3\pi}{4} + \sum_{k=1}^{n} \tan^{-1} \frac{1}{2^k} &> -\frac{3\pi}{4} + \sum_{k=1}^{3} \tan^{-1} \frac{1}{2^k} \approx -1.523 > -\pi/2, \\
-\frac{3\pi}{4} + \sum_{k=1}^{n} \tan^{-1} \frac{1}{2^k} &< -\frac{3\pi}{4} + \sum_{k=1}^{n} \frac{1}{2^k} < -\frac{3\pi}{4} + 1 < 0 < \pi/2.
\end{aligned}$$

So finally we have the correct limit:

$$\lim_{n\to\infty} \tan^{-1} u_n = -\frac{3\pi}{4} + \sum_{1}^{\infty} \tan^{-1} \frac{1}{2^n}. \tag{39}$$

4 In order to compute the value of this limit to two significant figures, one approach is just to sum enough terms in the series.

This gives

$$\begin{aligned}
&-2.35619449 + 0.46364761 + 0.24497866 + 0.12435499 + 0.06241881 + 0.03123983 + 0.01562373 \\
&\quad + 0.00781234 + 0.00390623 + 0.00195312 + 0.00097656 + \ldots \\
&\quad = -1.39928261\ldots
\end{aligned}$$

Up to two significant places this is -1.4, and up to 2 significant decimal places this is -1.40.

Remark The tables provided in the examination were probably insufficient to give the correct estimate to two significant decimal places.

Here is the method the examiners wanted: Let $e(x) = x - \tan^{-1} x$ be the error between $\tan^{-1} x$ and x for small x. Then

$$e(0) = 0, \quad e'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2}.$$

Since $e'(x) > 0$ if $x > 0$, it follows that $e(x) > 0$ for $x > 0$.

By the Intermediate Value Theorem, for each $x > 0$ there is some $\xi \in (0, x)$ such that

$$e(x) = e'(\xi)(x - 0) = \frac{\xi^2}{1+\xi^2}\, x < \xi^2 x < x^3. \tag{40}$$

This is good, because if x is small then x^3 is much smaller again.

From (39) and the definition of $e(x)$,

$$\begin{aligned}\lim_{n\to\infty}\tan^{-1}u_n &= -\frac{3\pi}{4}+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{2^2}+\sum_{n\geq 3}\frac{1}{2^n}+\sum_{n\geq 3}\left(\tan^{-1}\frac{1}{2^n}-\frac{1}{2^n}\right)\\ &= -2.35619449+0.46364761+0.24497866+.25-\sum_{n\geq 3}e(2^{-n})\\ &= -1.39756822-\sum_{n\geq 3}e(2^{-n})\end{aligned} \tag{41}$$

From (40),

$$\sum_{n\geq 3}e(2^{-n})\leq\sum_{n\geq 3}\frac{1}{2^{3n}}=\frac{1}{2^9}\frac{1}{1-1/8}=.00223\ldots.$$

So

$$\lim_{n\to\infty}\tan^{-1}u_n=-1.3975\pm.0023.$$

In particular,

$$\lim_{n\to\infty}\tan^{-1}u_n=-1.40 \tag{42}$$

to two significant decimal places. □

Tips & Tricks This question is a challenge!

- It follows from (39) that the sequence u_n defined recursively by (27) has the limit

$$u_n\to\tan\left(-\frac{3\pi}{4}+\sum_1^\infty\tan^{-1}\frac{1}{2^n}\right).$$

  involving tan and arctan $=\tan^{-1}$. This is certainly not obvious.
- The approach taken in part **2** is motivated by the "hint" to sketch $x=\tan y$, the given form of the limit, and knowing the formula for the tangent of the sum of two angles.
- The condition (34) necessary to obtain (33) was overlooked (surprisingly) by the examiners. For this reason the given limit was incorrect.
- If you use the recursive calculator Recursive Function Calculator you will find $u_{55}=\cdots=u_{60}=-5.7398165269881$ to 13 decimal places. In the calculator set

$$u_0=f(0)=1,\quad u_n=f(n)=(2^n*f(n-1)+1)/(2^n-f(n-1)).$$

  Then use the Arctan(x) Calculator to get $\tan^{-1}u_n=-1.39830603$ to 8 decimal places for $n=55,\ldots,60$.

  As a check, note that this result from using the two calculators is in agreement with (42), hence with (39) and so not with (28)!

Question 8

The focus of a parabola is the point (h, k) and its directrix is $y = d$. Show that its equation can be written in the form

$$2py = x^2 + 2qx + r$$

and express p, q, r in terms of h, k, d.

Write down the coordinates of the focus of the parabola

$$2py = x^2 + 2qx.$$

The two parabolas

$$2py = x^2 + 2qx$$
$$2p'x = y^2 + 2q'y$$

have the same focus. Prove that their tangents at the origin are inclined at 45°.

State this theorem in general terms (without reference to coordinates).

Solution.

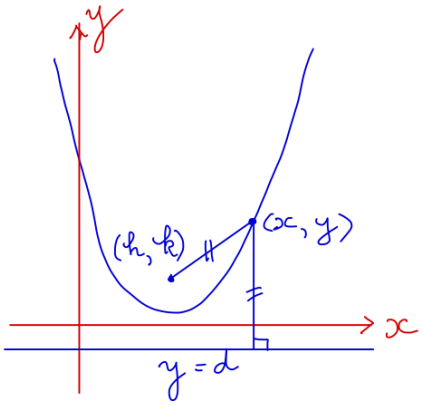


Figure 8: Distance from (x, y) to the focal point (h, k) and to the directrix $y = d$ are equal.

1 From the definition of focus and directrix, points (x, y) on the parabola are given by

$$(x - h)^2 + (y - k)^2 = (y - d)^2,$$

see Figure (8). That is,

$$x^2 - 2hx + (h^2 + k^2 - d^2) = 2(k - d)y.$$

This is in the form

$$2py = x^2 + 2qx + r, \tag{43}$$

where

$$p = k - d, \quad q = -h, \quad r = h^2 + k^2 - d^2. \tag{44}$$

2 For the parabola P given by

$$2py = x^2 + 2qx, \tag{45}$$

we have $r = 0$ in (43). It follows from (44) that

$$d^2 = h^2 + k^2 = q^2 + (p + d)^2 = q^2 + p^2 + 2pd + d^2,$$

and so

$$d = -\frac{p^2+q^2}{2p}, \quad p+d = \frac{p^2-q^2}{2p}.$$

Hence again from (44), the focus of P has coordinates

$$(h,k) = (-q, p+d) = \left(-q, \frac{p^2-q^2}{2p}\right). \tag{46}$$

3 Let P' be the parabola

$$2p'x = y^2 + 2q'y. \tag{47}$$

By the previous argument with the roles of x and y switched, the focus of P' has coordinates

$$(h',k') = \left(\frac{p'^2-q'^2}{2p'}, -q'\right). \tag{48}$$

Note that $x=0$, $y=0$ satisfies (45) and (47), and so the origin $(0,0)$ lies on both P and P'. Differentiating both (45) and (47) with respect to x, at the origin we find $\dfrac{dy}{dx} = \dfrac{q}{p}$ for P and $\dfrac{dy}{dx} = \dfrac{p'}{q'}$ for P'.

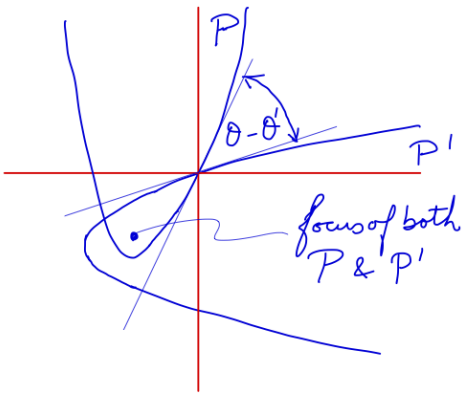


Figure 9: Parabolas with a common focus and orthogonal principal axes.

Let θ and θ' be the angles that the two tangents at the origin make with the x-axis. Then $\tan\theta = q/p$ and $\tan\theta' = p'/q'$. We want to show that $\theta - \theta' = \pm\pi/4$, or equivalently that $\tan(\theta-\theta') = \pm 1$. But

$$\tan(\theta-\theta') = \frac{\tan\theta - \tan\theta}{1+\tan\theta\tan\theta'} = \frac{q/p - p'/q'}{1+p'q/(pq')} = \frac{qq'-pp'}{pq'+p'q}.$$

So we want to show

$$\begin{aligned} pq'+p'q &= \pm(qq'-pp'), \\ \text{equivalently, } (pq'+p'q)^2 &= (qq'-pp')^2, \\ \text{equivalently, } 4pp'qq' &= (pp')^2+(qq')^2-(pq')^2-(p'q)^2. \end{aligned} \tag{49}$$

We are given that P and P' have the same focus, and so from (46) and (48),

$$\begin{aligned} -q &= \frac{p'^2-q'^2}{2p'}, \quad -q' = \frac{p^2-q^2}{2p}, \\ \therefore \quad -2p'q &= p'^2-q'^2, \quad -2pq' = p^2-q^2, \\ \therefore \quad 4pp'qq' &= (pp')^2+(qq')^2-(pq')^2-(p'q)^2. \end{aligned}$$

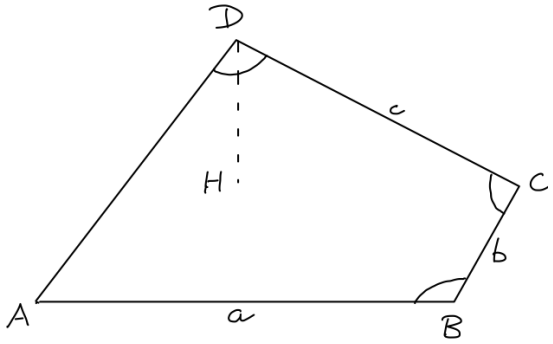
This establishes (49) and hence that the two tangents at the origin are inclined at 45°.

4 The general result is that if two parabolas have orthogonal axes and a common focus, then at each of the two points of intersection their tangents are inclined at 45° to each other.

□

Tips & Tricks The main point is to find an expression for the (tan of the) angle between the two tangent vectors. See (49) and the discussion which precedes it. The ± in the first line of (49) gives a hint we should square each side of the equality. Then we go back to (46) and (48) and think about how we can use the information there.

Question 9



In the diagram above, which represents a skew quadrilateral ABCD, the angles at B, C, D are right angles. If H is the foot of the perpendicular from D to the plane ABC, prove that HC is parallel to AB.

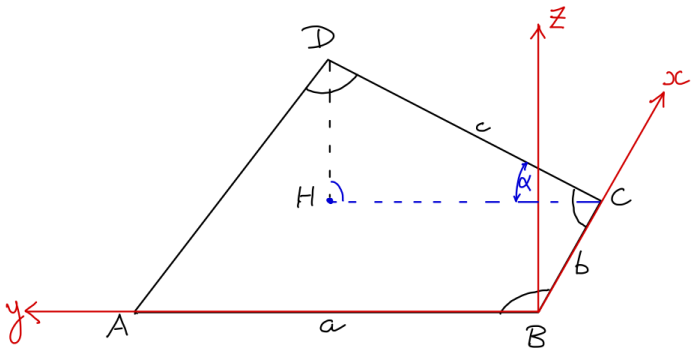
If the lengths of AB, BC, CD are respectively a, b, c, and CD makes an angle α with the plane ABC, prove that

$$c = a\cos\alpha \quad \text{and} \quad AD^2 = a^2 + b^2 - c^2.$$

Find

(a) the volume of the tetrahedron ABCD,
(b) the perpendicular distance of A from the plane BCD,
(c) the length of the orthogonal projection of CD on to AB,
(d) an expression for the angle between the planes DCA, DCB.

Solution.



Choose cartesian coordinates with origin at B, x-axis in direction BC, y-axis in direction BA and z-axis in the direction given by the right hand thumb rule corresponding to the x, y axes in that order.

Then

$$A = (0, a, 0), \quad B = (0, 0, 0), \quad C = (b, 0, 0), \quad D = (b, c\cos\alpha, c\sin\alpha), \quad H = (b, c\cos\alpha, 0) \tag{50}$$

It follows that

$$\begin{aligned} &\overrightarrow{BC} = (b, 0, 0), \quad \overrightarrow{BA} = (0, a, 0), \quad \overrightarrow{CD} = (0, c\cos\alpha, c\sin\alpha), \quad \overrightarrow{AD} = (b, c\cos\alpha - a, c\sin\alpha), \\ &\overrightarrow{CH} = (0, c\cos\alpha, 0), \quad \overrightarrow{HD} = (0, c\sin\alpha, 0), \end{aligned} \tag{51}$$

(i) HC is parallel to AB, since both are parallel to the y-axis from (51).

(ii) Since the angle at D is a right angle, $\overrightarrow{AD} \perp \overrightarrow{CD}$. Using (51),

$$\begin{aligned} 0 = \overrightarrow{AD} \cdot \overrightarrow{CD} &= c\cos\alpha(c\cos\alpha - a) + c^2\sin^2\alpha \\ &= c^2 - ac\cos\alpha = c(c - a\cos\alpha). \end{aligned}$$

Since $c \neq 0$ it follows that $c = a\cos\alpha$.

(iii) From (51)

$$\begin{aligned} AD^2 &= b^2 + (c\cos\alpha - a)^2 + c^2\sin^2\alpha \\ &= b^2 + c^2 - 2ac\cos\alpha + a^2 \\ &= a^2 + b^2 - c^2 \qquad \text{(using (ii))} \end{aligned}$$

(a) The volume of a tetrahedron is $\frac{1}{3}$ times the area of any side times the perpendicular distance from the opposite vertex to that side. (This is often expresssed as "$\frac{1}{3}$ times area of base times height").

Hence

$$\begin{aligned} \text{volume of tetrahedron} &= \frac{1}{3} \times \text{area}(ABC) \times \text{length}(HD) \\ &= \frac{1}{3} \cdot \frac{1}{2}ab \cdot c\sin\alpha = \frac{abc}{6}\sin\alpha. \end{aligned}$$

(b) Since the volume is also $\frac{1}{3}$ times (the area of BCD) times (the perpendicular distance of A from the plane BCD), it follows from (a) that

$$\frac{abc}{6}\sin\alpha = \frac{1}{3} \times \frac{1}{2}(bc) \times (\text{perp dist from } A \text{ to } BCD).$$

Hence the perpendicular distance from A to BCD is $a\sin\alpha$.

(c) The orthogonal projection of any point P onto AB is given by the y-coordinate of P. So from (50) the orthogonal projections of C and D are $(0,0,0)$ and $(0, c\cos\alpha, 0)$ respectively.

Hence the length of the orthogonal projection of CD onto AB is $c\cos\alpha$.

(d) The angle between two planes is the (acute) angle between their normals.

From (51), the upward pointing (by the right hand thumb rule) normal to DCA is given by

$$\begin{aligned} \overrightarrow{DA} \times \overrightarrow{DC} &= \begin{vmatrix} i & j & k \\ -b & a - c\cos\alpha & -c\sin\alpha \\ 0 & -c\cos\alpha & -c\sin\alpha \end{vmatrix} \\ &= i\begin{vmatrix} a - c\cos\alpha & -c\sin\alpha \\ -c\cos\alpha & -c\sin\alpha \end{vmatrix} - j\begin{vmatrix} -b & -c\sin\alpha \\ 0 & -c\sin\alpha \end{vmatrix} + k\begin{vmatrix} -b & a - c\cos\alpha \\ 0 & -c\cos\alpha \end{vmatrix} \\ &= (-ac\sin\alpha, -bc\sin\alpha, bc\cos\alpha). \end{aligned}$$

The squared length of this vector is

$$\begin{aligned} &a^2c^2\sin^2\alpha + b^2c^2\sin^2\alpha + b^2c^2\cos^2\alpha = a^2c^2\sin^2\alpha + b^2c^2 \\ &\quad = c^2(a^2\sin^2\alpha + b^2) = c^2(a^2 - a^2\cos^2\alpha + b^2) = c^2(a^2 + b^2 - c^2), \end{aligned}$$

where we used (ii) for the last equality. Hence the upward pointing unit length vector perpendicular to DCA is

$$\frac{1}{\sqrt{a^2 + b^2 - c^2}}(-a\sin\alpha, -b\sin\alpha, b\cos\alpha). \tag{52}$$

Similarly, the upward pointing normal to DCB is given by

$$\overrightarrow{CD} \times \overrightarrow{CB} = \begin{vmatrix} i & j & k \\ 0 & c\cos\alpha & c\sin\alpha \\ -b & 0 & 0 \end{vmatrix} = (0, -bc\sin\alpha, bc\cos\alpha).$$

This has length bc and so the upward pointing unit length vector perpendicular to DCB is

$$(0, -\sin\alpha, \cos\alpha). \tag{53}$$

The scalar product of the two unit length vectors in (52) and (53) is

$$\frac{b}{\sqrt{a^2+b^2-c^2}},$$

and so the angle between them, and hence between the planes DCA and DCB, is

$$\cos^{-1}\frac{b}{\sqrt{a^2+b^2-c^2}}.$$

□

Tips & Tricks Use Cartesian coordinates. Choose the origin and coordinate axes in a manner which simplifies the relevant expessions as much as possible. Another possibility would be to choose the origin at C instead of at B.

Question 10

Prove that if the lines $y = mx$, and $y = m'x$ harmonically separate $y = nx$, $y = n'x$, then

$$mm' - \frac{1}{2}(m + m')(n + n') + nn' = 0.$$

$y = mx$, $y = -mx$ are two given fixed lines, and H, (h, k), and H', (h', k') are two fixed points.

A point P moves so that the lines through P parallel to the fixed lines harmonically separate PH, PH'. Prove that the locus of P is a hyperbola, and find the coordinates of its centre, the equations of its principal axes, and its eccentricity.

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An Aside The notion of "harmonic separation" is not standard terminology, and the corresponding ideas are not now part of the high school curriculum, so here for context is the relevant background.

However, the solution of the question requires none of the following material other than an appropriate choice of line L in order to compute the cross ratio of the lines $y = mx$, $y = m'x$ and $y = nx$, $y = n'x$. See Figure 11 and the proof of (55).

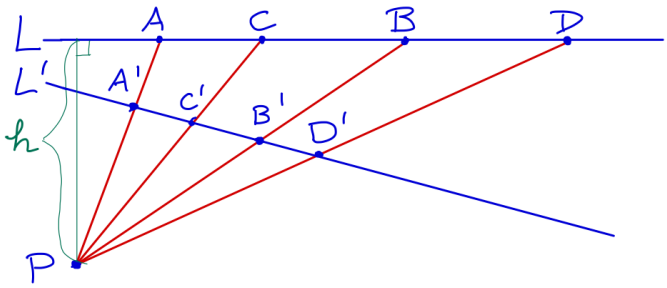

Figure 10: $\frac{AC}{BC} \Big/ \frac{AD}{BD} = \frac{A'C'}{B'C'} \Big/ \frac{A'D'}{B'D'} = \frac{\sin \angle APC}{\sin \angle BPC} \Big/ \frac{\sin \angle APD}{\sin \angle BPD}$

Cross Ratio Suppose the points A, B, C, D lie on a straight line L, as in Figure 10. The *cross ratio* $(AB : CD)$ of the points A, B, C, D taken in that order is defined by

$$(AB : CD) = \frac{AC}{BC} \Big/ \frac{AD}{BD} \,. \tag{54}$$

Here AC, BC, etc. are signed distances. For example, if the line L in Figure 10 is oriented from left to right then AC is positive and BC is negative.

Proposition: *Suppose P is a point not on L, L' is a second straight line not containing P, and L' crosses PA, PB, PC, PD at $A', B', C'D'$ respectively. Then the corresponding cross ratios for L and L' are equal:*

$$(AB : CD) = (A'B' : C'D'), \quad \textit{i.e.} \quad \frac{AC}{BC} \Big/ \frac{AD}{BD} = \frac{A'C'}{B'C'} \Big/ \frac{A'D'}{B'D'} \,.$$

Moreover, both equal

$$\frac{\sin \angle APC}{\sin \angle BPC} \Big/ \frac{\sin \angle APD}{\sin \angle BPD} \,.$$

Proof. Let h be the perpendicular distance from P to L, see Figure 10. Then

$$\begin{aligned}
\text{area } \triangle PAC &= \frac{1}{2}h \cdot AC = \frac{1}{2}PC \cdot PA \ \sin\angle APC \\
\text{area } \triangle PBC &= \frac{1}{2}h \cdot BC = \frac{1}{2}PC \cdot PB \ \sin\angle BPC \\
\text{area } \triangle PAD &= \frac{1}{2}h \cdot AD = \frac{1}{2}PD \cdot PA \ \sin\angle APD \\
\text{area } \triangle PBD &= \frac{1}{2}h \cdot BD = \frac{1}{2}PD \cdot PB \ \sin\angle BPD
\end{aligned}$$

(Here $\sin\angle APC$ is positive, $\sin\angle BPC$ is negative, etc.) Hence

$$\begin{aligned}
(AB:CD) &= \frac{AC}{BC} \Big/ \frac{AD}{BD} \\
&= \frac{PA \ \sin\angle APC}{PB \ \sin\angle BPC} \Big/ \frac{PA \ \sin\angle APD}{PB \ \sin\angle BPD} \\
&= \frac{\sin\angle APC}{\sin\angle BPC} \Big/ \frac{\sin\angle APD}{\sin\angle BPD} \ .
\end{aligned}$$

Therefore the cross ratio of the distances equals the "cross ratio" of the corresponding angles subtended at P. But these angles are the same for L and L'. Hence $(AB:CD) = (A'B':C'D')$. □

Definition: If A, B, C, D are on a straight line not containing P, then the points $\{A, B\}$ *harmonically separate* the points $\{C, D\}$, and the lines $\{PA, PB\}$ *harmonically separate* the lines $\{PC, PD\}$, if $(AB:CD) = -1$.

Remarks

- By the previous Proposition, one can use any line crossing all 4 lines $\{PA, PB, PC, PD\}$ and the corresponding intersection points, in order to compute the relevant cross ratio.
- The significance of $(AB:CD) = -1$ is that the points C and D divide the line segment AB internally and externally in the same ratio. The reason for -1 and not $+1$ is that in the $+1$ case it follows that $C = D$.

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Solution to Question.

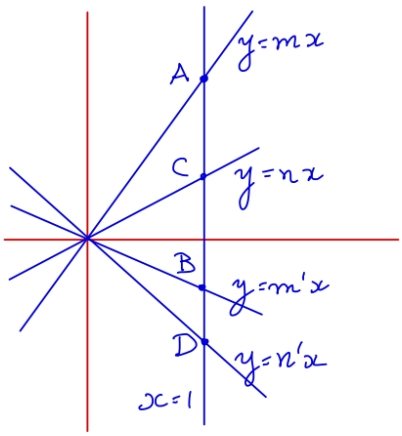

Figure 11: $y = mx$ and $y = m'x$ harmonically separate $y = nx$ and $y = n'x$.

See Figure 11. Consider the line $x = 1$. The lines $y = mx$, $y = m'x$, $y = nx$ and $y = n'x$, cross this line at the y-values m, m', n, n' respectively. Hence from (54) with $A = (1, m)$, $B = (1, m')$, $C = (1, n)$ and $D = (1, n')$, and using the definition of "harmonically separate",

$$\begin{aligned}
(AB : CD) = \frac{n-m}{n-m'} \Big/ \frac{n'-m}{n'-m'} &= -1 \\
\therefore \quad (n-m)(n'-m') &= -(n-m')(n'-m) \\
\therefore \quad 2mm' + 2nn' - (mn + mn' + m'n + m'n') &= 0 \\
\therefore \quad mm' - \frac{1}{2}(m+m')(n+n') + nn' &= 0. \qquad (55)
\end{aligned}$$

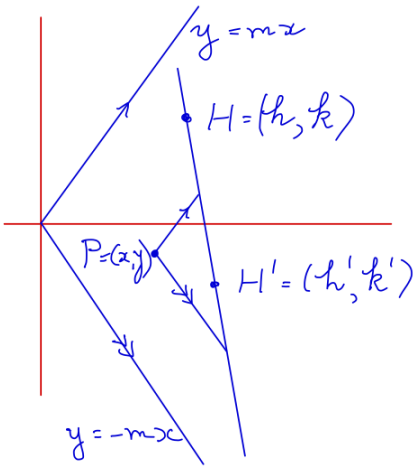

Figure 12: The lines through P with slope $\pm m$ harmonically separate PH and PH'.

See Figure 12. Let P have coordinates (x, y). Denote the slopes of the lines PH from P to H, and PH' from P to H', by

$$n = \frac{k-y}{h-x}, \quad n' = \frac{k'-y}{h'-x},$$

respectively. The slopes of the lines parallel to the fixed planes are m and $-m$ respectively.

We can apply (55) with m, m', n, n' there, replaced here by $m, -m, n, n'$ respectively. This is justified since translation, in this case moving P to the origin, does not change the slopes of the four lines through P. Note also from (54) that a pair of lines L_1, L_2 harmonically separate another pair L_3, L_4 if and only if L_3, L_4 harmonically separate L_1, L_2.

This gives

$$\begin{aligned}
-m^2 + \frac{k-y}{h-x} \times \frac{k'-y}{h'-x} &= 0 \\
\therefore \quad m^2(x-h)(x-h') &= (y-k)(y-k') \\
\therefore \quad m^2\left(x^2 - x(h+h') + hh'\right) &= y^2 - y(k+k') + kk' \\
\therefore \quad m^2\left(\left(x - \frac{h+h'}{2}\right)^2 - \left(\frac{h+h'}{2}\right)^2 + hh'\right) &= \left(y - \frac{k+k'}{2}\right)^2 - \left(\frac{k+k'}{2}\right)^2 + kk' \\
\therefore \quad m^2\left(x - \frac{h+h'}{2}\right)^2 - \left(y - \frac{k+k'}{2}\right)^2 &= \frac{m^2}{4}(h-h')^2 - \frac{1}{4}(k-k')^2. \qquad (56)
\end{aligned}$$

This is an hyperbola with centre $\big((h+h')/2, (k+k')/2\big)$, and principal axes $x = \frac{1}{2}(h+h')$ and $y = \frac{1}{2}(k+k')$.

The eccentricity of the hyperbola in the *standard form* $x^2/a^2 - y^2/b^2 = 1$ is $\sqrt{1 + b^2/a^2}$. In order to put (56) into standard form, let

$$\gamma = \frac{m^2}{4}(h-h')^2 - \frac{1}{4}(k-k')^2.$$

We assume $\gamma \neq 0$, since $\gamma = 0$ implies $\dfrac{k-k'}{h-h'} = \pm m$, which implies HH' is parallel to one of the two fixed lines, and so the locus of P is not properly defined.

If $\gamma > 0$, then on dividing both sides of (56) by γ and translating $\big((h+h')/2, (k+k')\big)$ to the origin, the locus of P is in standard form with $a = \sqrt{\gamma}\, m^{-1}$ and $b = \sqrt{\gamma}$. In this case the eccentricity is $\sqrt{1+b^2/a^2} = \sqrt{1+m^2}$. Note that translating to the origin does not change the geometric properties such as eccentricity.

If $\gamma < 0$ then, on dividing by $-\gamma$ and again translating $\big((h+h')/2, (k+k')\big)$ to the origin, the locus of P is $y^2/(-\gamma) - m^2x^2/(-\gamma) = 1$. Switching x and y amounts to reflection in the line $x = y$ and so does not affect the eccentricity. So the eccentricity is $\sqrt{1 + \dfrac{(-\gamma)m^{-2}}{-\gamma}} = \sqrt{1+m^{-2}}$.

□

Tips & Tricks

- In order to derive (55) you need to know the meaning of "harmonically separates". But then the derivation is straightforward.
- The next point is that you should treat P as the origin in order to derive (56). This is legitimate since the cross ratio is invariant under translation. *Why?*
- Finally, you need to separately consider the cases $\gamma > 0$ and $\gamma < 0$ in order to derive the eccentricity.