NEW SOUTH WALES

Department of Education

LEAVING CERTIFICATE EXAMINATION, 1962

Mathematics I

HONOURS PAPER

Chief Examiner: Professor T. G. ROOM, Sc.D.
Assessors: H. MULHALL, B.Sc., Ph.D.
R. J. SMITH, Ph.D., M.Sc.

Time allowed—Three hours

Candidates may submit answers to *eight* questions only.

The questions are of equal value.

Marks will not be awarded to answers where the work is not shown.

Marks will be deducted for careless or badly arranged work.

Mathematical tables and squared paper will be provided.

Answers to the two parts of this paper are to be returned in *separate books* marked Part A or Part B.

The division of the paper into Parts A and B does not restrict a candidate's choice of questions.

Part A

1. (i) Show that $a + b + c$ is one factor of

$$\begin{vmatrix} a - b - c & 2a & 2a \\ 2b & b - c - a & 2b \\ 2c & 2c & c - a - b \end{vmatrix}$$

and find the value of the determinant in its simplest form.

(ii) Show that the equations

$$\begin{aligned} x + y + 2z &= 2 \\ 2x - y + 3z &= 2 \\ 5x - y + az &= 6, \end{aligned}$$

have a unique solution if a is not equal to 8. Find *all* the solutions if $a = 8$.

2. (i) Show that if m and n are positive integers and $m \neq n$,

$$\int_0^{\pi} \cos mx \cos nx \, dx = 0.$$

What is the value of this integral when $m = n$?

(ii) Differentiate $\sin^{n-1}\theta \cos\theta$ and express the result in terms of $\sin\theta$ only.

Deduce that

$$\int_0^{\pi/2} \sin^n\theta d\theta = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}\theta d\theta.$$

Evaluate $\displaystyle\int_0^{\pi/2} \sin^6\theta d\theta$.

3. Find the indefinite integrals

(i) $\displaystyle\int x^2e^{-x}dx$,

(ii) $\displaystyle\int \frac{7+x}{1+x+x^2+x^3}\,dx$,

(iii) $\displaystyle\int \frac{e^{-2x}}{e^{-x}+1}\,dx$.

4. (i) Sketch the graph of the function e^{kx}, where k is a positive constant.

Find the equation of the tangent to the curve $y = e^{kx}$ which passes through the origin. Deduce that the equation

$$e^{kx} = x$$

has 0, 1 or 2 real roots according as k is greater than, equal to or less than $1/e$.

(ii) Show that if $|x| < 1$,

$$e^{-x} - \frac{1-x}{(1-x^2)^{\frac{1}{2}}(1-x^3)^{\frac{1}{3}}} = ax^5 + bx^6 + \dots,$$

where a and b are constants. Find the value of a.

5. (i) Show that the greatest value taken by the function $1 + 2x - e^x$ is 0·386, to three decimal places.

(ii) Prove that the volume, V, the area of curved surface, S, and the radius of the base, r, of a right circular cone are connected by the equation

$$9V^2 = r^2(S^2 - \pi^2r^4).$$

Show that the maximum volume for a given area of curved surface S, is

$$\frac{2^{\frac{1}{2}}S^{\frac{3}{2}}}{\pi^{\frac{1}{2}}3^{\frac{7}{4}}}.$$

Part B

6. Sketch, referred to the same axes, the curves

$$y = \frac{a^3}{x^2+a^2}$$

$$y = \frac{ax^2}{x^2+a^2}$$

where a is a positive constant. The finite area between these curves is rotated

(i) about the x-axis,

(ii) about the y-axis.

Find, in each case, the volume generated.

7. (i) If $z = x + iy$, and

$$w = \frac{z-1}{z},$$

express w in the form $u + iv$ where u and v are real. Hence show that if $|z| = 1$, then $|w - 1| = 1$.

(ii) Give a geometrical description of the locus of the point in the Argand diagram representing the complex number z which satisfies the condition

$$|z + i| = |z + 3 + 4i|.$$

(iii) Find the greatest value of arg z, when

$$|z - i| = \tfrac{1}{2}.$$

(iv) Z_1, Z_2 are the points in the Argand diagram representing the complex numbers z_1, z_2, where

$$z_2 = \frac{1}{z_1 - 4} + 2.$$

What is the locus of Z_2 as Z_1 describes the circle centre 4 and radius 3?

8. For the curve given by the parametric equations

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta),$$

where a is a positive constant, find the equation of the tangent at the point θ and show that the normal at this point has the equation

$$x \sin\frac{\theta}{2} + y \cos\frac{\theta}{2} = a\theta \sin\frac{\theta}{2}.$$

P is the point θ and Q is the point $\theta + \Delta\theta$ on the above curve; R is the point of intersection of the normals at P and Q. Show that as $\Delta\theta$ tends to 0 the limiting position of R is the point

$$\{a(\theta + \sin\theta), a(\cos\theta - 1)\}.$$

9. (i) By considering a definite integral as the limit of a sum, show that

$$\lim_{n \to \infty} \left(\frac{1}{n} + \frac{1}{n+1} \cdots + \frac{1}{2n}\right) = \int_0^1 \frac{dx}{1+x}$$

and give the value of this limit correct to three decimal places.

(ii) By means of a freehand sketch, show that if $f(x)$ is a positive function which is steadily *decreasing* as x increases, then

$$0 < \sum_{r=1}^{n} f(r) - \int_1^{n+1} f(x)dx < f(1).$$

Deduce that

$$0 < 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log(n+1) < 1.$$

10. (i) Write down the first four terms and a general term of the series for e^x.

Show that

$$\sum_1^{\infty} \frac{3n - 1}{n!} = 3\sum_1^{\infty} \frac{1}{(n-1)!} - \sum_1^{\infty} \frac{1}{n!}$$

and deduce the sum of the infinite series on the left.

(ii) State the *comparison test* for the convergence of a series of positive terms. Discuss the convergence of the series having general term—

(*a*) $u_n = \frac{1}{n} \sin \frac{\pi}{\sqrt{n}}$

(*b*) $u_n = \frac{x^n}{n(1 + x^{2n})}$, for positive values of x.