NEW SOUTH WALES

Department of Education

LEAVING CERTIFICATE EXAMINATION, 1962

Mathematics II

HONOURS PAPER

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Time allowed—Three hours

Candidates may submit answers to *eight* questions only.

The questions are of equal value.

Marks will not be awarded to answers where the work is not shown.

Marks will be deducted for careless or badly arranged work.

Mathematical tables and squared paper will be provided.

1. (i) Prove that

$$\left(1 + \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n = -2^n \cos^n \frac{\pi}{n}.$$

(ii) Prove that, if n is any integer, positive or negative,

$$(-1 + i\sqrt{3})^n + (-1 - i\sqrt{3})^n$$

has either the value 2^{n+1} or the value -2^n.

2. (i) Draw sketches to show the loci represented on the Argand diagram by

(*a*) $|z - a| = k$,

(*b*) $\arg(z - a) = \alpha$,

(*c*) $|z - a| + |z - b| = c$.

where a, b are complex constants, k, c, α are real constants, and $c > |a - b|$.

(ii) Prove that, if $z \neq 0$,

(a) $u = z + \dfrac{|z|^2}{z}$ is always real,

(b) $v = \dfrac{|z| - iz}{|z| + iz}$ is always pure imaginary, provided the real part of z is not zero, and that, if $\arg z = \theta$,

$$v = -i(\sec\theta + \tan\theta).$$

3. F is the point $(ae, 0)$ and d is the line $x = a/e$ $(e > 1)$. M is the foot of the perpendicular from a variable point P to d, and P moves so that

$$FP^2 = e^2PM^2.$$

Find the equation of the locus.

Draw a sketch showing clearly the principal axes of the curve, the foci, the directrices and the asymptotes, marking on each of them its equation or coordinates.

Express e in terms of α, the angle between the asymptotes.

A line, parallel to one asymptote, is drawn through a focus F of a hyperbola. It meets the hyperbola in H, the directrix corresponding to F in D, the other asymptote in K and the conjugate axis in R. Prove that

$$FH = HD,$$
$$FK = KR.$$

4. Express $x^n - 1$ as a product of real linear or quadratic factors, distinguishing the cases n odd and n even.

Thence or otherwise prove that

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\left(\frac{n-1}{2n}\pi\right) = \sqrt{\left(\frac{n}{2^{n-1}}\right)}$$

when n is odd, and find the corresponding result when n is even.

5. Draw a sketch to show the relation between the point $(a\cos\varphi,\ b\sin\varphi)$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the corresponding point on the auxiliary circle.

Write down the equation of the circle on which the points (x_1, y_1), (x_2, y_2) are the extremities of a diameter.

P is a variable point on an ellipse of which F is one focus. Prove that the circle on PF as diameter touches the auxiliary circle of the ellipse.

6. Prove that the feet of the four normals to the ellipse, $E = 0$, where $E \equiv x^2/a^2 + y^2/b^2 - 1$, from the point (x_1, y_1) are the intersections of $E = 0$ with the rectangular hyperbola $H_1 = 0$, where

$$H_1 \equiv \left(\frac{1}{b^2} - \frac{1}{a^2}\right) xy + \frac{y_1}{a^2}\, x - \frac{x_1}{b^2}\, y.$$

Give a reason why, if $\lambda H_1 + E = 0$ represents a line pair, then this line pair is one of the three which join pairs of the feet of the four normals.

The join of the feet of two of these four normals is the line

$$\frac{lx}{a} + \frac{my}{b} + 1 = 0;$$

prove that the join of the other two feet is

$$\frac{x}{la} + \frac{y}{mb} - 1 = 0.$$

Find the values of x_1, y_1 in terms of l, m.

7. Sketch the graph of the function

$$x = \tan y$$

and indicate how a range of principal values of the inverse tangent should be chosen. Denoting the principal value of the inverse tangent by $\tan^{-1}x$, give a formula for the general value of the inverse tangent.

If a sequence is defined by

$$u_0 = 1$$

$$u_{n+1} = \frac{2^{n+1}u_n + 1}{2^{n+1} - u_n}$$

show that $\lim\limits_{n \to \infty} \tan^{-1}u_n = \frac{\pi}{4} + \sum\limits_{1}^{\infty} \tan^{-1}\frac{1}{2^n}$.

Using the tables compute the value of this limit to two significant figures.

8. The focus of a parabola is the point (h,k) and its directrix is $y = d$. Show that its equation can be written in the form

$$2py = x^2 + 2qx + r$$

and express p, q, r in terms of h, k, d.

Write down the coordinates of the focus of the parabola

$$2py = x^2 + 2qx.$$

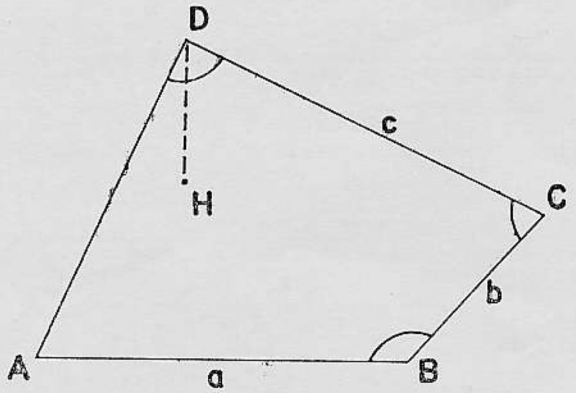
The two parabolas

$$2py = x^2 + 2qx$$
$$2p'x = y^2 + 2q'y$$

have the same focus. Prove that their tangents at the origin are inclined at 45°.

State this theorem in general terms (without reference to coordinates).

9.



In the diagram above, which represents a skew quadrilateral ABCD, the angles at B, C, D are right angles. If H is the foot of the perpendicular from D to the plane ABC, prove that HC is parallel to AB.

If the lengths of AB, BC, CD are respectively a, b, c, and CD makes an angle α with the plane ABC, prove that

$$c = a \cos \alpha \text{ and } AD^2 = a^2 + b^2 - c^2.$$

Find

(*a*) the volume of the tetrahedron ABCD,

(*b*) the perpendicular distance of A from the plane BCD,

(*c*) the length of the orthogonal projection of CD on to AB,

(*d*) an expression for the angle between the planes DCA, DCB.

10. Prove that if the lines $y = mx$, $y = m'x$ harmonically separate $y = nx$, $y = n'x$, then

$$mm' - \tfrac{1}{2}(m + m')(n + n') + nn' = 0.$$

$y = mx$, $y = -mx$ are two given fixed lines, and H, (h,k), and H$'$, (h',k') are two fixed points.

A point P moves so that the lines through P parallel to the fixed lines harmonically separate PH, PH$'$. Prove that the locus of P is a hyperbola, and find the coordinates of its centre, the equations of its principal axes, and its eccentricity.