SYDNEY UNIVERSITY MATHEMATICAL SOCIETY COMPETITION

Solution should be handed in to Mrs. Kovacs, Room 805, by 1 p.m. on Tuesday, 12th May.

Students who have completed second year mathematics are restricted to Questions 7 to 12.

1. Let $f(x)$ be a function of x for $x \geq 0$ such that for every integer $k > 0$

$$f(x+k) = f(x)$$

and

$$\int_0^1 f(x)\,dx = 0 .$$

Prove there exists $c \geq 0$ such that for all $x \geq c$

$$\int_c^x f(t)\,dt \geq 0 .$$

2. The following game is played by two players A and B. $2n+1$ matches are placed on a table. A and B take turns in taking away a number of matches, the numbers allowed being 1, 2 or 3 until all matches are removed. The player left with an odd number of matches wins. A makes the first move.

Prove that if neither player makes a mistake then

(i) if $n \neq$ ~~$4k + 1$~~ $4k+2$ then A wins;

(ii) if $n =$ ~~$4k + 1$~~ $4k+2$ then B wins.

Work out the winning strategies.

3. Let $x_1, \ldots, x_n$ be positive numbers such that

$$x_1 + x_2 + \ldots + x_n = n .$$

Prove that

$$\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} \geq n ,$$

and equality holds only if

$$x_1 = x_2 = \ldots = x_n = 1 .$$

4. The number of odd binomial coefficients in a given binomial expansion is a power of 2.

5. Find all solutions of the simultaneous equations

$$a^3 - b^3 - c^3 = 3abc$$

$$a^2 = 2(b+c)$$

in positive (non zero) integers.

6. Given a convex n-sided polygon with k diagonals such that every pair of these diagonals meet, prove that $k \leq n$.

7. Let A', B', C' be points on the sides BC, CA, AB of the triangle ABC. Prove that one of the triangles AB'C', CA'B', BC'A' has an area no greater than the area of A'B'C'.

8. Let c_n denote the n-th integer which can be written in the form k^l $(k,l = 2,3,4,\ldots)$. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{c_n - 1} = 1 .$$

9. If the associative law is not valid then the product $a_1a_2a_3$ has two possible values, viz. $a_1(a_2a_3)$, $(a_1a_2)a_3$. Similarly $a_1a_2a_3a_4$ has five possible values, viz. $a_1(a_2(a_3a_4))$, $a_1((a_2a_3)a_4)$, $(a_1a_2)(a_3a_4)$, $((a_1a_2)a_3)a_4$, $(a_1(a_2a_3))a_4$.
How many possible values has the product

$$a_1a_2\ldots a_n$$

for general n?

10. Let $x_1, x_2, \ldots$ be a sequence of distinct numbers in the interval (0,1) such that if a and b are two distinct numbers in the interval (0,1) then there exists an x_n between a and b. The points $x_1, \ldots, x_{n-1}$ divide the interval into n disjoint subintervals and x_n divides one of these subintervals into two parts. Let a_n, b_n be the lengths of these two parts. Prove that

$$\sum_{n=1}^{\infty} a_nb_n(a_n+b_n) = 1/3 .$$

11. If $n = 4m + 3$, determine n subsets $S_1, S_2, \ldots, S_n$ of the set $\{1, 2, \ldots, n\}$ such that (i) each S_i is of order 2m+1 (i.e., contains 2m+1 elements); (ii) for $i \neq j$ $S_i \cap S_j$ is of order m.
Partial solutions (i.e. for many but not all values of n) will be accepted.

12. The Cayley-Hamilton theorem states that every matrix satisfies its characteristic equation. Prove that if

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ - & - & - \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

and $A_{ij} = a_{ij}I$ where I is the n × n identity matrix, then the determinant of the following $n^2 \times n^2$ matrix is identically zero:

$$\begin{pmatrix} A_{11}-A & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22}-A & \cdots & A_{2n} \\ - & - & - & - \\ A_{n1} & A_{n2} & \cdots & A_{nn}-A \end{pmatrix} .$$

For example if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\begin{vmatrix} 0 & -b & b & 0 \\ -c & a-d & 0 & b \\ c & 0 & d-a & -b \\ 0 & c & -c & 0 \end{vmatrix} = 0 .$$

SOLUTIONS OF SUMS

Competition problems.

PART I

1. Let $F(x) = \int_0^x f(t)dt$. At some $x = c$, $0 \le c \le 1$, $F(x)$ has a minimum value, $F(x) \ge F(c)$ for $0 \le x \le 1$. Now $F(0) = F(1) = 0$, and by the assumed periodicity of $f(x)$, $F(k+y) = F(y)$ for every integer $k \ge 0$. In particular, $F(k) = 0$ and $F(x) = \int_0^{[x]} f + \int_{[x]}^{x} f = F(x-[x]) \ge F(c)$ for every $x \ge 0$ since $0 \le x - [x] \le 1$. Hence

$$F(x) - F(c) = \int_c^x f \ge 0 \qquad \text{for all } x \ge 0 .$$

Solved by J.E. Hutchinson, I. Peterson, P.W. Donovan.

2. Solution by Hutchinson:

A "strategic position" P_k for player X is defined as a position in which (1) X's opponent has the next move, (2) there are 4k or 4k+1 matches left on the table, (3) X has an even number of matches in hand if k is odd and an odd number if k is even.

First we show that whatever is Y's (X's opponent) next move, X can reply so as to move into a strategic position P_{k-1}.

(i) If Y takes 1 or 3, X replies by taking 3 or 1 respectively.

(ii) If Y takes 2, X replies by taking 1 or 3 matches depending on whether 4k or 4k+1 matches were originally on the table.

Clearly the moves result in a P_{k-1} position for X. Finally, there will result a position P_0 for X with 0 or 1 matches on the table (in the latter case Y's turn to move) and X holding an odd number of matches.

If the game commenced with 8k+1, A takes 1 match; if it commenced with 8k+3, A takes 3; in both cases a P_{2k} results. If the game commenced with 8k+7, A takes 2 matches and produces P_{2k+1}. In all these cases A wins. Finally if the game commenced with 8k+5, B is in a strategic P_{2k+1} and so wins.

Also solved by Peterson.

3. $a + \frac{1}{a} \ge 2$ for $a > 0$, equality only holding if $a = 1$. Hence

$$(x_1+\ldots+x_n) + \left(\frac{1}{x_1} + \ldots + \frac{1}{x_n}\right) = \sum_i \left(x_i + \frac{1}{x_i}\right) \ge 2n ,$$

$$\sum_i \frac{1}{x_i} \ge 2n - n = n .$$

Equality holds only if $x_1 = x_2 = \ldots = x_n = 1$.

Solved by Hutchinson, Peterson; Donovan.

4. Solution by Hutchinson:

First we show (A): $\binom{n}{r}$ is even if $n = 2^p$ and $0 < r < n$.

For $p = 1$ the result is trivially true; assume result for some $p \geq 1$ and let $0 < s \leq 2^p$. Then

$$\binom{2^{p+1}}{s} = \binom{2^p}{0}\binom{2^p}{s} + \binom{2^p}{1}\binom{2^p}{s-1} + \dots + \binom{2^p}{s}\binom{2^p}{0}$$

$$= 2\binom{2^p}{s} + \sum_{i=1}^{s-1}\binom{2^p}{i}\binom{2^p}{s-i}$$

which is even by the induction hypothesis. By symmetry it is also even for $2^p < s < 2^{p+1}$.

Now let $n = 2^p + t$, $0 < t < 2^p$ and consider $\binom{n}{r}$.

(a) Take $0 < r \leq t$, then

$$\binom{n}{r} = \binom{2^p}{0}\binom{t}{r} + \sum_{i=1}^{r}\binom{2^p}{i}\binom{t}{r-i} \equiv \binom{t}{r} \pmod 2 \quad \text{by (A).}$$

Consequently $\binom{2^p+t}{r}, \binom{2^p+t}{2^p+r}$ and $\binom{t}{r}$ are all odd or all even.

(b) Take $t < r < 2^p$, then $\binom{n}{r} = \sum_{i=0}^{t}\binom{t}{i}\binom{2^p}{r-i}$ which is even by (A).

By (a) and (b) the number of odd binomial coefficients in the expansion of $(1+x)^{2^p+t}$ is twice the number in the expansion of $(1+x)^t$ $(0 < t < 2^p)$. Repeatedly reducing the index of expansion until it is equal to a power of 2 it is seen that the number of odd coefficients in the expansion $(1+x)^n$ is equal to 2^α where α is the number of steps in the reduction. The latter is equal to the number of unit digits in the binary representation of n.

4. Generalization by Peterson: The number of binomial coefficients in $(1+x)^n$ not divisible by a given prime p is equal to $\prod_{i=0}^{k}(1+a_i)$ where

$$n = a_0 + a_1 p + \dots + a_k p^k, \quad 0 \leq a_i < p.$$

<u>Proof</u>: Let $0 < r < p^k$ and let p^d be the largest power of p which goes into r. Then $p^{k-d} \mid \binom{p^k}{r}$. For $r = 1$ the statement is obvious; for $r \geq 1$ it follows by induction from $\binom{p^k}{r+1} = \binom{p^k}{r}\frac{p^k - r}{r+1}$. In particular, all binomial coefficients $\binom{p^k}{r}$, $0 < r < p^k$, are divisible by p, and $(1+x)^{p^k} \equiv (1+x^{p^k}) \pmod p$. Therefore

$$(1+x)^{mp^k} \equiv (1+x^{p^k})^m$$

$$\equiv \sum_{r=0}^{m}\binom{m}{r}x^{rp^k} \pmod p;$$

hence if $0 < m < p$, $(1+x)^{mp^k}$ has exactly $m+1$ coefficients not divisible by p .

Generally for $0 \leq j < p^k$, $0 < m < p$,

$$(1+x)^{mp^k+j} = (1+x)^{mp^k}(1+x)^j = (1+x)^j \sum_{r=0}^{m} \binom{m}{r} x^{rp^m} \pmod{p} .$$

Therefore if there are $N(j)$ coefficients in $(1+x)^j$ not divisible by p then there are $(m+1)N(j)$ such coefficients in $(1+x)^{mp^k+j}$, and the result is evidently true also for $m = 0$. This gives

$$N(mp^k+j) = (m+1)N(j) \quad \text{for } 0 \leq j < p^k , \quad 0 \leq m < p).$$

in particular $N(n) = (a_k+1)\, N(a_0+a_1p+\ldots+a_{k-1}p^{k-1})$

which gives the required result, by induction on k .

Q5

5. $a^3-b^3-c^3-3abc = (a-b-c)(a^2+b^2+c^2+ab-bc+ca) = 0$, hence either $a - b - c = 0$ or $a^2+b^2+c^2+ab-bc+ca = 0$. But

$$2(a^2+b^2+c^2+ab-bc+ca) = (a+b)^2 + (b-c)^2 + (c+a)^2 = 0$$

requires $a = -b = -c$ which is impossible if a,b,c are positive.

Hence $a = b+c$, and from $a^2 = 2(b+c)$

$$a = 2, \ b = 1, \ c = 1 .$$

Solved by T. Dent, Hutchinson, Peterson; Donovan.

6. Let $P_1, P_2, \ldots, P_n$ be the polygon. If there is a vertex with only one given diagonal running from it, we can reduce n by omitting this vertex and its diagonal. The remaining polygon has $k-1 \leq n-1$ diagonals by induction on n . (For $n \leq 5$ the theorem is obviously true as the total number of possible diagonals is $\leq n$).

Assume therefore that each vertex has at least two diagonals running out of it. Let P_1P_i, P_1P_j $(i < j)$ be the two outermost diagonals from P_1 . All vertices P_k with $i < k < j$ can only be connected with P_1 ; for if P_hP_k is a diagonal and $1 < h < j$ then P_hP_k does not meet P_1P_j and if $h > j$ then it does not meet P_1P_i . Hence there is no vertex between P_i and P_j and we only have to consider the case when exactly two diagonals meet at each vertex. In this case the number of diagonals is $\frac{2n}{2} = n$.

Solved by Hutchinson.

Let $P_{\alpha_i} P_{\beta_i}$ $(1 \le \beta_i < \alpha_i \le n)$, $i = 1,\ldots,k$ be the given diagonals, arranged so that $1 \le \alpha_1 \le \alpha_2 \le \ldots \le \alpha_k \le n$. For simplicity we denote them by (α_i, β_i). Then for every j, $\beta_j \le \alpha_1$ since otherwise (α_1, β_1) and (α_j, β_j) had no point in common. Furthermore $\beta_i \ge \beta_j$ for $i > j$ since otherwise (α_i, β_i) and (α_j, β_j) had no point in common. (We use here the fact that the polygon is convex.) We now have the inequalities

$$1 \le \beta_1 \le \ldots \le \beta_k \le \alpha_1 \le \ldots \le \alpha_k \le n,$$

where $\beta_i < \beta_{i+1}$ if $\alpha_i = \alpha_{i+1}$ and $\alpha_i < \alpha_{i+1}$ if $\beta_i = \beta_{i+1}$.

Let m_1 be the number of values i for which $\alpha_i < \alpha_{i+1}$,

m_2 the number of values i for which $\beta_i < \beta_{i+1}$.

Then $m_2 \ge k - 1 - m_1$, $m_1 + m_2 \ge k-1$ (1).

On the other hand $m_2 < \beta_k$, $m_1 \le n - \alpha_1$, hence by $\alpha_1 \ge \beta_k$,

$m_2 < n - m_1$, $m_1 + m_2 \le n - 1$. (2).

(1) and (2) give $k-1 \le n-1$, $k \le n$.

PART 2.

7. Let $BA'/BC = \alpha$, $CB'/CA = \beta$, $AC'/AB = \gamma$, $0 \le \alpha, \beta, \gamma \le 1$.

(Remark by Hutchinson: if A', B', C' are not between B and C; C and A; A and B respectively then question is incorrect.)

Let area of ΔABC be one unit; then

Area $\Delta AB'C' = (1-\beta)\gamma$, $\Delta BC'A' = (1-\gamma)\alpha$, $\Delta CA'B' = (1-\alpha)\beta$,

Area $\Delta A'B'C' = 1-(1-\beta)\gamma - (1-\gamma)\alpha - (1-\alpha)\beta$

$= (1-\alpha)(1-\beta)(1-\gamma) + \alpha\beta\gamma$.

Now if $\alpha > \frac{1}{2}, \beta < \frac{1}{2}$ then $(1-\alpha)\beta < (1-\alpha)(1-\beta)$ and $(1-\alpha)\beta < \alpha\beta$ hence $(1-\alpha)\beta < (1-\alpha)(1-\beta)(1-\gamma) + \alpha\beta\gamma$,

$$\Delta CA'B' < \Delta A'B'C'.$$

Similarly, if $\beta > \frac{1}{2}$, $\gamma < \frac{1}{2}$ then $\Delta AB'C' < \Delta A'B'C'$

and if $\gamma > \frac{1}{2}$, $\alpha < \frac{1}{2}$ then $\Delta BC'A' < \Delta A'B'C'$.

Hence we may assume either $\alpha \le \frac{1}{2}, \beta \le \frac{1}{2}, \gamma \le \frac{1}{2}$ or $\alpha \ge \frac{1}{2}, \beta \ge \frac{1}{2}, \gamma \ge \frac{1}{2}$.

We show that $\Delta A'B'C' \ge \frac{1}{4}$ which will prove our statement. Suppose that $\alpha = \frac{1}{2} + a$, $\beta = \frac{1}{2} + b$, $\gamma = \frac{1}{2} + c$ where either $0 \le a,b,c \le \frac{1}{2}$ or $-\frac{1}{2} \le a,b,c \le 0$. Then

$$\Delta A'B'C' = (\tfrac{1}{2}+a)(\tfrac{1}{2}+b)(\tfrac{1}{2}+c) + (\tfrac{1}{2}-a)(\tfrac{1}{2}-b)(\tfrac{1}{2}-c)$$
$$= \frac{1}{4} + ab + ac + bc \ge \frac{1}{4}, \text{ q.e.d.}$$

Solved by {Hutchinson, Donovan,

9. Solution by Donovan:

Let $T(n)$ be the number of possible values of the product with n-factors. In any product the final multiplication is of the form $[a_1 \ldots a_r][a_{r+1} \ldots a_n]$, $1 \leq r < n$. The first factor can be evaluated in $T(r)$ ways, the second (independently) in $T(n-r)$ ways.

Hence
$$T(n) = \sum_{r=1}^{n-1} T(r)T(n-r), \quad T(1) = 1 \qquad (1)$$

Now write
$$F(x) = \sum_{n=1}^{\infty} T(n)x^n .$$

Then
$$(F(x))^2 = \sum_{n=2}^{\infty} \left(\sum_{r=1}^{n-1} T(r)T(n-r)\right)x^n = \sum_{n=2}^{\infty} T(n)x^n$$
$$= F(x) - x$$

by (1) hence $F(x) = \frac{1}{2}(1-\sqrt{1-4x}) = \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1}x^n$

by the well known binomial expansion. Hence
$$T(n) = \frac{1}{n}\binom{2n-2}{n-1} .$$

10. If at the n-th stage (i.e. after x_n has been inserted) the $n+1$ subintervals are $d_{no}, d_{n1}, \ldots, d_{nn}$ then $\sum_{k=1}^{n} a_k b_k (a_k + b_k) = \frac{1}{3}(1 - \sum_{i=o}^{n} d_{ni}^{\,3})$.

This follows immediately from
$$3a_n b_n (a_n + b_n) = (a_n + b_n)^3 - a_n^{\,3} - b_n^{\,3} .$$

We have to show that $\lim_{n \to \infty} \left(\sum_{i=o}^{n} d_{ni}^{\,3} \right) = 0$.

Now if $\Delta_n = \max\{d_{ni} ; \; i = 0,1,\ldots,n\}$

then $\sum_{i=o}^{n} d_{ni}^{\,3} \leq \Delta n^2 \sum_{i=o}^{n} d_{ni} = \Delta n^2$

since $\sum_{i=o}^{n} d_{ni} = 1$. But $\lim_{n \to \infty} \Delta_n = 0$ because of the denseness of the sequence x_n .

Solution by K. Price:

Consider a right square pyramid P_o of unit height on unit base A . The perpendicular from the vertex to the base of the pyramid is of unit length and represents the interval $[0,1]$. Choose a point x_1 on the perpendicular, cutting it into two lengths a_1, b_1 (a_1 top, b_1 bottom). Draw the horizontal plane through x_1 and the four vertical planes through the lines where the horizontal plane cuts the slant sides of the pyramid. These planes will cut the pyramid in ten pieces as follows:

1) A top pyramid P_1 similar to P_o, with height a_1.

2) A rectangular prism with base a_1^2 and height b_1, i.e. volume $a_1^2 b_1$.

3) Four side pieces of base $\frac{1}{2}a_1(1-a_1)$ and altitude b_1, of volume $\frac{1}{4}a_1(1-a_1)b_1 = \frac{1}{4}a_1 b_1^2$ each, hence total volume $a_1 b_1^2$.

4) Four corner pieces which pushed together give a pyramid P_2 similar to P_o, with height b_1. So the selection of x_1 divides the pyramid into the volume $a_1^2 b_1 + a_1 b_1^2 = a_1 b_1 (a_1 + b_1)$ and two pyramids of measure a_1 and b_1 respectively. Clearly the selection of x_n will extract from one of the n pyramids obtained previously a volume $a_n b_n (a_n + b_n)$ and leave two pyramids corresponding to the division. But the total volume of the n pyramids tend to 0 because of the denseness of the points x_i, and so

$$\sum_1^{\infty} a_n b_n (a_n + b_n) = \text{volume of } P_o = \frac{1}{3}.$$

Also solved by Donovan.

11. The problem is unsolved for general $n = 4m + 3$. If $n = p$ is a prime number, a solution is obtained as follows:

Let $x_1 \equiv 1^2$, $x_2 \equiv 2^2, \ldots, x_{2m+1} \equiv (2m+1)^2$ be the quadratic residues modulo p. Set

$$S_i = (i+x_1, i+x_2, \ldots, i+x_{2m+1})$$

where $i + x_k$ is replaced by $i+x_k-p$ if $i+x_k > p$. Then the subsets S_i satisfy both conditions (i) and (ii). The first is trivial, the second follows from the fact that for given residue $a \equiv j-i$, the congruence

$$k^2 - l^2 \equiv a \pmod p), \quad 1 \le k,l \le 2m+1$$

has exactly m distinct pairs (k,l) for solution.

For instance if $m = 1$, $n = 7$, the following sets supply a solution: (124) (235) (346) (457) (156) (267) (137).

This is the well known model of a finite projective plane with 7 points.

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8. We can write C_n uniquely in the form $C_n = k^l$ where $k \ge 2$ is not a power and $l \ge 2$. Then

$$\sum_{n=1}^{\infty} \frac{1}{C_n - 1} = \sum_{k=2}^{\infty}{}' \sum_{l=2}^{\infty} \frac{1}{k^l - 1} = \sum_{k=2}^{\infty}{}' \sum_{l=2}^{\infty} \sum_{p=1}^{\infty} \frac{1}{k^{lp}} = \sum_{k=2}^{\infty}{}' \sum_{p=1}^{\infty} \sum_{l=2}^{\infty} \frac{1}{k^{lp}}$$

$$= \sum_{k=2}^{\infty} \sum_{p=1}^{\infty} \frac{1}{k^p(k^p - 1)}.$$

(see p7)

where Σ' indicates that the sum is extended for all $k \ge 2$ which is

12. Solution by Donovan.

If $M = (m_{ij})$ is an $n \times n$ matrix, denote by $E(M)$, $D(M)$ the following $n^2 \times n^2$ matrices:

$$E(M) = \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \cdots & & \\ M_{n1} & \cdots & M_{nn} \end{pmatrix}, \quad D(M) = \begin{pmatrix} M & 0 & \cdots & 0 \\ 0 & M & & \\ \vdots & & & \\ 0 & \cdots & & M \end{pmatrix}$$

where $M_{ij} = m_{ij} I$.

The following is easily verified:

$$E(A)E(B) = E(AB), \quad D(A)D(B) = D(AB) \tag{1}$$

$$E(A)D(B) = D(B)E(A) \, , \tag{2}$$

$$E(H^{-1}) = \{E(H)\}^{-1} \, , \quad D(H^{-1}) = \{D(H)\}^{-1} \tag{3}$$

for non-singular H . We want to show that

$$\det\,(E(A) - D(A)) = 0 \, .$$

Now if H is non-singular then

$$E(H^{-1}AH) - D(H^{-1}AH) \quad \text{and} \quad E(A) - D(A)$$

are similar; for

$$E(H^{-1}AH) = D(H^{-1})E(H^{-1})E(A)E(H)D(H)$$

$$D(H^{-1}AH) = D(H^{-1})E(H^{-1})D(A)E(H)D(H)$$

by (1) and (2) .

We may assume that the field of the a_{ij} is algebraically closed. Determine H so that $B = H^{-1}AH$ is upper triangular, with the characteristic roots $\lambda_1, \ldots, \lambda_n$ in the diagonal. Then $E(B) - D(B)$ itself is upper triangular and the characteristic equation is

$$\prod_{i=1}^{n} \prod_{j=1}^{n} (\lambda - \lambda_i + \lambda_j) = \lambda^n \prod_{i \neq j} (\lambda - \lambda_i + \lambda_j) = 0 \, .$$

But by the previous result $E(A) - D(A)$ has the same characteristic equation, which shows not only that $\det(E(A) - D(A)) = 0$ but that $E(A) - D(A)$ has rank $\leq n^2 - n$.

8.(cont.) not a perfect power. Reordering of summation is permissible since all terms are positive, provided that the last sum converges. But every integer $m \geq 2$ can uniquely be written in the form k^p, $p \geq 2$, k (≥ 2) not a power. Hence the last sum is equal to

$$\sum_{m=2}^{\infty} \frac{1}{m(m-1)} = \sum_{m=2}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m}\right) = 1 \, .$$

Solved by Donovan, Hutchinson.