1965

SYDNEY UNIVERSITY MATHEMATICAL SOCIETY

Second Annual Problem Competition

This competition is open to all undergraduates. Separate prizes to the value of £5. 0. 0 will be awarded for the best solution from each of the four years. The competition closes on Tuesday, 29th June, 1965, and all solutions must be in the hands of the Pure Mathematics Secretaries at the Universities of Sydney or New South Wales by this date. Contestants must supply their full name, term address, 'phone number, University, Faculty and year.

Contestants are allowed to make free use of textbooks or any other legitimate source of information. Partial answers will be accepted, particularly from the first and second year students. Credit will be given for elegance of solutions and for possible extensions and generalizations, e.g., if you can deal with Question 9 when n is even, etc.

The prizes will be presented at a special S.U.M.S. address at 5.15 p.m. on Tuesday, July 6th, 1965. This will be followed by Supper in the Tea Room of the Pure Mathematics Department at Sydney University, to which the Mathematics Staffs of both Sydney's Universities will be invited.

The Committee of S.U.M.S. would like to express its gratitude to Professor Szekeres, Professor of Pure Mathematics at the University of New South Wales, for editing the problems, and to Mathematics Staffs of both Universities for kindly donating the prize-money.

1. If $\sum_{k=1}^{n} p_k = 1$, $\sum_{k=1}^{n} q_k = 1$, $p_k > 0$, $q_k > 0$, prove that

$$\sum_{k=1}^{n} q_k \log (q_k/p_k) \geq 0.$$

2. A matrix has m rows and n columns, $m \geq n$. All columns of the matrix are distinct. Show that one row can be removed such that the columns are still distinct. Is the condition $m \geq n$ necessary?

3. A function $f(x)$ is analytic on the real line and has the property that for each real x there is a rational number $r = r(x) \neq 0$ such that

$$f(x) = f(x+r).$$

Prove that $f(x)$ is periodic. Produce a counter-example to show that it is not sufficient merely to assume continuity of $f(x)$.

4. Determine all the formal power series

$$\sum_{n=0}^{\infty} a_n z^n$$

with the property that for $k = 1, 2, \ldots$, the equation

$$\sum_{n=0}^{k} a_n z^n = 0$$

has all its roots on the unit circle.

5. $p_1, \ldots, p_n$ are points on the unit circle; d_{ij} is the distance $\overline{p_i p_j}$. Prove that

$$\sum_{1 \le i < j \le n} d_{ij}^{\,2} \le n^2$$

For what sets $\{p_i\}$, $i = 1, \ldots, n$, does equality hold?

Establish a corresponding result for points on the unit 3-sphere.

6. A positive sequence $\{a_n\}$ is such that $a_{n-1} \le a_n + a_n^2$. Prove that $\sum a_n$ diverges (i) if the sequence is monotone decreasing (i.e. $i < j$ implies $a_i \geq a_j$), (ii) without any such restriction.

7. Construct a positive sequence $\{a_n\}$ such that

$$a_{n-1} = a_n + a_n^2 \quad \text{for all } n.$$

8. A sequence of primes

$$p_1 = 2,\ p_2 = 3,\ p_3 = 7,\ p_4 = 43,\ \ldots$$

is such that p_{n+1} is the largest prime factor of $p_1 p_2 \ldots p_n + 1$. Prove that 5 and 11 are not members of the sequence. (It is not known whether the sequence is monotone increasing.)

9. Let n be odd and S_n the symmetric group of n objects, consisting of all permutations of these objects. Prove that to a given $\pi \in S_n$, not the identity, one can find a second permutation $\sigma \in S_n$ such that π and σ together generate the whole group, S_n.

10. Let $(x_1, x_2, \ldots, x_n)$ be an arbitrary permutation of the set of integers $(1, 2, \ldots, n)$.

$$U(\underline{x}) = U(x_1, \ldots, x_n) = \sum_{i=1}^{n-1} | x_{i+1} - x_i | .$$

E_n is the expected value of $U(\underline{x})$, as $(x_1, \ldots, x_n)$ varies over all $n!$ permutations of $(1, \ldots, n)$.

Prove that

$$E_n = \tfrac{1}{3}(n-1)(n+1) .$$

Determine the variance of $U(\underline{x})$.

$\frac{1}{90}(n+1)(4n-7)(n-2)$

SECOND S.U.M.S. COMPETITION, 1965.

SOLUTIONS.

1. First Solution (A.J. Berrick. S.U., and N. Larschy, N.S.W.).

$$\sum_{k=1}^{n} q_k \log \frac{q_k}{p_k} \geq \sum_{k=1}^{n} q_k \left(\frac{q_k - p_k}{q_k}\right) = \sum_{k=1}^{n} (q_k - p_k)$$

$$= 1 - 1 = 0$$

(Using the inequality $-\log(1-x) \geq x$ for $-1 < x < 1$)

Equality only when $p_k = q_k$ for every k.

Second Solution (C.J. Smyth, A.N.U.)

Use the well known inequality between the weighted arithmetic and geometric means of positive numbers:

$$\frac{\Sigma a_i b_i}{\Sigma b_i} \geq [\Pi a_i^{b_i}]^{1/\Sigma b_i}$$

with $a_i = p_i/q_i$ and $b_i = q_i$: Gives

$$\Pi \left(\frac{q_i}{p_i}\right)^{q_i} \geq 1, \quad \Sigma q_i \log (q_i/p_i) \geq 0.$$

Third Solution (several solvers).

For fixed $q_1, \ldots, q_n$, consider the p_k as variables, subject to the constraint $\Sigma p_k = 1$. By Lagrange, the condition for a critical point is

$$\frac{\partial}{\partial p_k} (\Sigma q_i \log (q_i/p_i)) + \lambda \frac{\partial}{\partial p_k} (\Sigma p_i) = 0, \quad k = 1, \ldots, n,$$

which gives

$$\frac{q_k}{p_k} + \lambda = 0, \quad \Sigma q_k = -\lambda \Sigma p_k, \quad \lambda = -1,$$

$p_k = q_k$ for every k. This is the only critical point, and a true maximum (the expression becomes $-\infty$ on the boundary of the region

$$\Sigma p_k = 1, \quad p_k > 0 \ (k = 1, \ldots, n).)$$

2. Solution by D.R. McKenzie, N.S.W.

Suppose that the matrix A has n distinct columns, and has the property that removal of any of its m rows will leave at least one pair of equal columns. We want to show that $m < n$.

Suppose that removal of the i - th row will cause the p_i - th and q_i - th columns to become equal (if there are several such pairs of columns, select any one of them). Thus we have selected m pairs of columns, (p_i, q_i), $i = 1, \ldots, m$, one for each row. The pairs (p_i, q_i) are all distinct, otherwise we had two equal columns already in A, contrary to assumption.

Now represent the columns of A by vertices of a graph (numbered $1, 2, \ldots, n$) and connect two vertices p and q by an edge if and only if the pair (p, q) appears among the pairs (p_i, q_i), $i = 1, \ldots, m$. The graph G so obtained contains no closed circuit. For if $(a_1, a_2) = (p_{i_1}, q_{i_1})$, $(a_2, a_3) = (p_{i_2}, q_{i_2}), \ldots, (a_r, a_{r+1}) = (p_{i_r}, q_{i_r})$ are edges of G and j is distinct from $i_1, i_2, \ldots, i_r$, then the j -th row has equal entries in the a_1-th, a_2-th, ..., a_{r+1}-th columns and therefore certainly $(p_j, q_j) \neq (a_1, a_{r+1})$.

The result $m < n$ now follows from the observation that a graph with n vertices which contains no closed circuit cannot have more than $n - 1$ edges.

Counter example for $m = n - 1$: $A_{ii} = 1$, $i = 1, \ldots, n - 1$,

$A_{ij} = 0$ for $i \neq j$.

Rows of A are clearly distinct, but removal of the i-th row will leave the i-th and n-th columns equal (both are 0).

3. Solution by J.E. Hutchinson (N.S.W.) R.H. Street, (Sydney) and P. Wark (N.S.W.)

For any rational number r, let S_r be the set of points in the closed interval $I = (x;\ 0 \le x \le 1)$ for which $f(x) = f(x+r)$.

Since the set of rational numbers is countable and by assumption $I = \bigcup_r S_r$, there is an r such that S_r contains infinitely many points. By Bobzane - Weierstrass, S_r has a limit point ξ in I.

Let $\{X_n\}$; $n = 1,2,\ldots$ be a sequence in S_r with $\lim_{n\to\infty} X_n = \xi$.

By assumption

$$f(X_n+r) = f(X_n), \quad n = 1,2,\ldots$$

But if $f(x)$ is analytic then so is $f(x+r)$ and therefore by the identity theorem of analytic functions $f(x+r) = f(x)$ in a neighbourhood of ξ.

Hence $f(x+r) = f(x)$ for all x.

Counterexample in the continuous case:

The function

$$f(x) = \sin 2\pi x \quad \text{for } -1\le x\le 1$$
$$f(x) = 0 \quad \text{for } |x|>1$$

is continuous and

$$f(x) = f(x-1) \quad \text{for } x\le -1 \text{ and } 0\le x\le 1,$$
$$f(x) = f(x+1) \quad \text{for } -1\le x\le 0 \text{ and } x\ge 1 .$$

Thus $f(x)$ satisfies the requirements but clearly it is not periodic.

4. (Several solvers).

If C is any non-zero constant and $|\beta| = 1$ then

$$C(1+\beta z + \beta^2 z^2 + \ldots + \beta^n z^n + \ldots)$$

has the required property. We show that there are no others.

Let then $f(z) = a_o+a_1 z+ \ldots +a_n z^n+\ldots$

be a formal power series with the required property. Since $a_o+a_1 z = 0$ has its roots on the unit circle, we must have $|a_o/a_1| = 1$ hence $a_o\ne 0$. We may assume $a_o = a_1 = 1$; otherwise consider $g(z) = \frac{1}{a_o} f\left(\frac{a_o}{a_1} z\right) = 1 + z +\ldots+ b_n z^n +\ldots$ instead of $f(z)$ which clearly has the same property.

We want to show that $b_n = 1$ for $n = 2,3,\ldots$.

Suppose that the statement is not true and $n(>1)$ is the smallest index with $b_n \neq 1$. Then

$$b_n z^n + z^{n-1} + \ldots + z + 1 = 0$$

has all its roots on the unit circle. The product of these roots is $(-1)^n / b_n$, therefore $|b_n| = 1$ and

$$b_n z^n = -\frac{z^n - 1}{z-1}, \quad \left|\frac{z^n-1}{z-1}\right| = 1, \quad |z^n - 1| = |z-1|$$

for all roots. Since both z and z^n are on the unit circle, this is only possible if either $z^n = z$ or $z^n = \bar{z}$.

In the first case $z^{n-1} = 1$, $1 + z + \ldots + z^{n-2} = 0$, $z^{n-1} + b_n z^n = 0$, $z\, b_n + 1 = 0$ for all roots which is impossible. Therefore $z^n = \bar{z} = z^{-1}$, $z^{n+1} = 1$, $1 + z + \ldots + z^n = 0$, $b_n = 1$.

5. <u>Solution</u> by J. Hutchinson (N.S.W.) and J. Warmsley (Newcastle)

Let $\underline{p}_1, \ldots, \underline{p}_n$ be vectors of unit length in Euclidean k - space, representing points on the (k - dimensional) unit sphere. Then $d_{ij} = |\underline{p}_i - \underline{p}_j|$ and

$$\sum_{i,j} d_{ij}^2 = \sum_{i,j} (\underline{p}_i - \underline{p}_j) \cdot (\underline{p}_i - \underline{p}_j)$$

$$= \sum_{i,j} (|\underline{p}_i|^2 + |\underline{p}_j|^2) - 2 \sum_{i,j} \underline{p}_i \cdot \underline{p}_j$$

$$= 2n^2 - 2 \sum_{i,j} \underline{p}_i \cdot \underline{p}_j$$

since $|\underline{p}_i|^2 = 1$. But

$$\sum_{i,j} \underline{p}_i \cdot \underline{p}_j = (\underline{p}_1 + \ldots + \underline{p}_n) \cdot (\underline{p}_1 + \ldots + \underline{p}_n) \geq 0$$

therefore

$$\sum_{i,j} d_{ij}^2 \leq 2n^2, \quad \sum_{1 \leq i < j \leq n} d_{ij}^2 \leq n^2,$$

5. Solution (continued).

equality holding whenever

$$\underline{p}_1 + \dots + \underline{p}_n = 0 .$$

The result is valid in any Euclidean k-space.

McKenzie noted that if n is even and with each point also the diametrically opposite point is in the given set, then Pythagoras gives immediately $\Sigma d_{ij}^{\ 2} = n^2$.

6. (After J. Hutchinson (N.S.W.) and R.H. Street, Sydney).

Suppose that a_n is monotone and Σa_n converges. Then we must have $na_n \to 0$ and $\exists\ m > 0$ so that setting

$(m - 1).\ a_{m-1} = c,$

we have $na_n < c$ for $n \geq m$. But then

$$\frac{c}{m-1} = a_{m-1} \leq a_m + a_{m^2} \leq a_{m+1} + a_{m^2} + a_{(m+1)^2}$$

$$\dots \leq a_{2m} + a_{m^2} + a_{(m+1)^2} + \dots + a_{4m^2}$$

$$< \frac{c}{2m} + \frac{c}{m^2} + \dots + \frac{c}{4m^2}$$

$$< \frac{c}{2m} + c\int_{m-1}^{2n} \frac{dx}{x^2} = \frac{c}{m-1} ,$$

a contradiction.

No correct solution was offered for the non-monotonic case.

7. Solution by Hutchinson and Smyth.

set tentatively

$$a_n = \sum_{r=1}^{\infty} b_r\, n^{-r} , \quad b_1 = 1.$$

Then $a_{n-1} = a_n + a_{n^2}$ gives

$$\sum_{k=1}^{\infty} b_k/n^k(1 - \frac{1}{n})^k = \sum_{k=1}^{\infty} b_k\, n^{-k}(1 + \frac{k}{n} + \frac{k(k+1)}{2!\,n^2} + \dots)$$

7. <u>Solution (Continued).</u>

$$= \sum_{r=1}^{\infty} (b_r n^{-r} + b_r n^{-2r}),$$

hence

$$\sum_{k=1}^{r} \binom{r}{k-1} b_k = 0 \quad \text{if} \quad r \quad \text{even}$$

$$= b_{\frac{1}{2}(r+1)} \quad \text{if} \quad r \quad \text{odd.}$$

Or, setting $kb_k = c_k$, the recursion is

$$\sum_{k=1}^{r-1} \binom{r}{k} c_k = 0 \quad \text{if} \quad r \quad \text{is odd}$$

$$= 2c_m \text{ if } r = 2m ,$$

with $c_1 = 1$.

Solving for the first few values of c_k we calculate

$$c_1 = 1, \; c_2 = -1, \; c_3 = 0, \; c_4 = 1, \; c_5 = -1, \; c_6 = 0 , \ldots$$

which suggests

$$c_k = 1 \qquad \text{if} \quad k \equiv 1 \pmod 3$$

$$c_k = -1 \qquad \text{if} \quad k \equiv -1 \pmod 3$$

$$c_k = 0 \qquad \text{if} \quad k \equiv 0 \pmod 3).$$

Assuming for the moment that this is true we obtain

$$\sum_{k=1}^{\infty} c_k x^{k-1} = \frac{1}{1 - x^3} - \frac{x}{1 - x^3} = \frac{1}{1+x+x^2}$$

and

$$\sum_{k=1}^{\infty} b_k x^k = \sum_{k=1}^{\infty} \frac{1}{k} c_k x^k = \int_0^x \frac{dt}{1+t+t^2}$$

hence

$$a_n = \sum_{k=1}^{\infty} b_k n^{-k} = \int_n^{\infty} \frac{dt}{1+t+t^2} = \frac{2}{\sqrt{3}} \text{ arcot } \frac{2}{\sqrt{3}} \; (n+\tfrac{1}{2}).$$

7. Solution (Continued).

Now irrespective of the assumptions made in the derivation it can be verified a posteriori that

$$a_n = c \text{ arcot } \frac{2}{\sqrt{3}}(n+\tfrac{1}{2}), \quad c > 0$$

does in fact satisfy the required equation. For

$$a_{n-1} - a_n = c\,[\text{arot } \frac{2}{\sqrt{3}}(n-\tfrac{1}{2}) - \text{arcot } \frac{2}{\sqrt{3}}(n+\tfrac{1}{2})]$$

$$= c \text{ arcot } \frac{2}{\sqrt{3}}(n^2+\tfrac{1}{2}) = a_{n^2}\,.$$

8. Solution by Smyth, Street and Anne Julienne (Sydney), simplified.

If 5 is a member of the sequence then $p_1p_2 \ldots p_n + 1$ (for some n) must be a power of 5, $p_1p_2 \ldots p_n = 5^k - 1$, which is impossible since the left hand side is $\equiv 2 \pmod 4$ (every p_n with $n > 1$ is odd) while the right hand side is divisible by 4.

Suppose then that 11 is a member, then $p_1p_2 \ldots p_n + 1$ for some n has 11 for its largest prime factor. But is is relatively prime to $p_1=2$, $p_2=3$ and $p_3=7$, therefore it must have the form 5^r11^s,

$$p_1p_2 \ldots p_n \div 5^r 11^s - 1.$$

The left hand side is $\equiv 2 \pmod 4$, the right hand side is $\equiv (-1)^s - 1 \pmod 4$ since $5 \equiv -1 \pmod 3$ therefore

$5^r11^s - 1 \equiv (-1)^{r+s} - 1 \pmod 3$.

Since the left hand side is divisible by 3, r+s must be even, and r is odd.

Now $11 \equiv 5^2 \pmod 7$ hence

$$5^r 11^s \equiv 5^{2r+s} \pmod 7$$

and we must have $5^{2r+s} \equiv 1 \pmod 7$.

8. Solution (Continued).

But 5 is a primitive root mod 7 (i.e. the lowest positive power of 5 which is $\equiv 1 \pmod 7$ is 5^6) therefore no odd power of 5 can be $\equiv 1 \pmod 7$, a contradiction.

9. This theorem is due to Sophie Picard. No correct solution was offered by competitors.

Denote by $x_1, x_2, \ldots, x_n$ the objects to be permuted. The usual cyclic notation will be used. We show that already a permutation of the type $\sigma = (12)(34\ldots n)$ can be found to generate S_n. Write $\tau = (12)$, $\rho = (34\ldots n)$, so that

$\rho\tau = \tau\rho = \sigma$.

First we note that $\sigma^{n-2} = \tau$, $\sigma^{n-1} = \rho$, since n is odd and $\rho^{n-2} = 1$. Hence σ itself generates a 2-cycle τ and a (disjoint) $\overline{n-2}$ -cycle ρ.

We distinguish two cases.

(i) π leaves at least one x_i unchanged. Without loss of generality we may assume that $\pi(x_1) = x_1$. Since π is not the identity, there is an i such that $\pi(x_i) \neq x_i$. Assume $\pi(x_2) = x_3$ and take σ as above. Then reading from right to left

$$\tau' = \pi\tau\pi^{-1} = (13) \quad \text{and } \rho^k\tau'\rho^{-k} = (1,\overline{3+k}),$$
$$k = 1,2,\ldots,n-3.$$

Thus all transpositions (12, (13),..., (1n) are in the group generated by σ and π, and therefore it is the whole of S_n.

(ii) π moves every x_i. Then π is not the product of disjoint 2-cycles such as (12,(34)(56)..., since n is odd, and therefore it must contain a cycle with more than 2 elements, e.g. $\pi = (123\ldots)(\ldots)\ldots$ But then $\tau' = \pi\tau\pi^{-1} = (23)$, $\rho^k\tau'\rho^{-k} = (2,\overline{3+k})$, $k = 1,2,\ldots,n-3$ and S_n is generated as before.

10. <u>Solution</u> by Hutchinson.

$$E_n = \frac{1}{n!} \sum_{\underline{x}} \sum_{i=1}^{n-1} |X_{i+1} - X_i|$$

where $\sum_{\underline{x}}$ runs through all permutations $\underline{x} = (x_1, x_2, \ldots, x_n)$.

There are $n!\,(n-1)$ terms, and each of the $\frac{1}{2}n(n-1)$ pairs (j,k) with $1 \le k < j \le n$ appears the same number of times, therefore

$$E_n = \frac{1}{n!} \; \frac{n!\,(n-1)}{n(n-1)} \; 2 \sum_{n \geq j > k \geq 1} (j-k)$$

$$= \frac{2}{n} \sum_{k=1}^{n-1} k(n-k) = \frac{1}{3}(n-1)(n+1).$$

To find $V[U(\underline{x})] = E[U(\underline{x})]^2 - \{E[U(\underline{x})]\}^2$:

(P) $$n!\; E[U(\underline{x})]^2 = \sum_{\underline{x}} \sum_{i,j} |x_{j+1} - x_j|\,|x_{i+1} - x_i| \,.$$

Consider $\Sigma|i-j|\,|k-\ell|$ where i,j,k,ℓ take all values $1,\ldots,n$.

There are n^4 terms. We take the following disjoint classes:

(1) Zero terms: $i=j$, $k=\ell$, or both, $2n^3-n^2$ cases.

(2) Non-zero terms in which $i=k$, $j=\ell$ or $i=\ell$, $j=k$, $2n(n-1)$ cases.

(3) Non-zero terms in which the pairs (i,j), (k,ℓ) have exactly one common member, e.g. $i=k$, $j=\ell$.
$4n(n-1)(n-2)$ cases

(4) Non-zero terms in which the pairs (i,j), (k,ℓ) have no common element, $n(n-1)(n-2)(n-3)$ cases.

Now $$\sum_{(1)} |i-j|\,|k-\ell| = 0 \qquad \text{(A)}$$

$$\sum_{(2)} |i-j|\,|k-\ell| = 2 \sum_{i,j} (i-j)^2$$

$$= \frac{n^2}{3}(n+1)(n-1) \qquad \text{(B)}$$

10. Solution (Continued).

$$\sum_{(3)} |i-g||k-\ell| = 4 \sum_{i\neq k} \sum_{j} |i-j||k-j|$$

$$= 4 \sum_{j} [\sum_{i} |i-j|].[\sum_{k} |k-j|] - 4 \sum_{i,j} |i-j|^2$$

$$= \frac{n}{15} (n+1)\ (n-1)\ (n-2)\ (7n+4) \qquad \text{(C)}$$

eventually. (all sums here are standard combinatorial expressions).

$$\sum_{(4)} |i-j||k-\ell| = \sum_{i,j,k,\ell} |i-j||k-\ell| - \{\sum_{(1)} + \sum_{(2)} + \sum_{(3)}\}$$

$$= \frac{n}{45} (n-1)\ (n+1)\ (n-2)\ (n-3)\ (5n+4) \qquad \text{(D)}$$

Returning to (P), there are $n!(n-1)$ terms $|x_{i+1}-x_i|^2$, therefore by (2) and (B) the corresponding sum is

$$\frac{n!(n-1)}{2n(n-1)} \frac{n^2}{3} (n+1)\ (n-1) = \frac{n!}{6} n(n-1)\ (n+1).$$

There are $n!2(n-2)$ terms $|x_{j+1}-x_j||x_{i+1}-x_i|$ with $j=i+1$ or $i=j+1$, therefore by (3) and (C) the corresponding sum is

$$\frac{n!.2(n-2)}{4n(n-1)(n-2)} \frac{n}{15} (n+1\ (n-1)\ (n-2)\ (7n+4)$$

$$= \frac{n!}{30} (n+1)\ (n-2)\ (7n+4).$$

There are $n!\ (n-2)\ (n-3)$ terms with $j\neq i+1$ and $i\neq j+1$, therefore by (4) and (D), the corresponding sum is

$$\frac{n!(n-2)(n-3)}{n(n-1)(n-2)(n-3)} \frac{n}{45} (n-1)\ (n+1)\ (n-2)\ (n-3)\ (5n+4).$$

$$= \frac{n!}{45} (n+1)(n-2)(n-3)(5n+4).$$

Adding all contributions, the result is

$$\frac{n!}{90} (n+1)\ (10n^3) - 6n^2 - 25n + 24),$$

and

$$V[U(\underline{x})] = E[U(\underline{x})]^2 - \{E[U(\underline{x})]\}^2 = \frac{1}{90} (n+1)(n-2)(4n-7)$$